Lecture Notes on the Theory of Elliptic Partial Differential Equations

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INTRODUCTION

We list here some of the material whose knowledge is assumed in this course.

1. Definition of a Banach space.

Linear space: Have $x+y$ and $c \cdot x$ (c a number) defined for all x,y in the space with

A. (i) $x+y = y+x$, (ii) $(x+y)+z = x+(y+z)$, (iii) $\exists$ a $\theta \ni x+\theta = x$ for all x, (iv) given x $\exists$ $y \ni x+y = \theta$

B. (v) $(cd) \cdot x = c \cdot x + d \cdot x$, (vi) $c(x+y) = c \cdot x + c \cdot y$, (vii) $c \cdot (d \cdot x) = (cd) \cdot x$, (viii) $1 \cdot x = x$.

THEOREM 1: θ is unique, $-x$ is unique, $0 \cdot x = \theta$, $(-1) \cdot x = -x$, $-(-x) = x$, $c \cdot \theta = \theta$, if $c \cdot x = \theta$ and $c \neq 0$, then $x = \theta$; if $c \cdot x = \theta$ and $x \neq \theta$, then $c = 0$.

DEFINITION: A Banach space is a complete linear metric space $\rho(x,y) = \rho(x-y, \theta)$ and $\rho(c \cdot x, \theta) = |c| \cdot \rho(x, \theta)$. The distance $\rho(x, \theta)$ is called the norm of x and is denoted by $\| x \|$.

THEOREM 2: (a) $\| x \| \geq 0$, the equality holding $\Longleftrightarrow x = \theta$

(b) $\| c \cdot x \| = |c| \cdot \| x \|$, (c) $\|x+y\| \leq \| x \| + \| y \|$.

(d) If E is a linear space with a norm defined on it satisfying (a), (b), (c), and we define $\rho(x,y) = \| x-y \|$, then ρ is a metric and E is a Banach space.

EXAMPLES: C_S, M_S, $L_p(\mu, S)$, (S a set, μ a measure).

2. Linear transformations, operators and functionals.

DEFINITIONS: Linear manifold, closed linear manifold, linear function,

linear transformation, (linear) operator, linear functional.

THEOREM 1: Suppose T is a linear transformation whose domain is the whole space and suppose T is continuous at one point. Then T is an operator.

THEOREM 2: If T is an operator, there is an M such that $\| T(x) \| \leq M \cdot \| x \|$ for all x.

DEFINITION: If T is an operator, $\| T \| = \sup_{x \neq \Theta} \dfrac{\| T(x) \|}{\| x \|}$.

THEOREM 3: Suppose each T_n is an operator (from B to B_1) and $T_n(x)$ converges (in B_1) to something for each x. If we define T by the equation $T(x) = \lim_{n \to \infty} T_n(x)$, then T is an operator, $\| T_n \|$ is uniformly bounded, and

$$\| T \| \leq \lim_{n \to \infty} \inf \| T_n \| .$$

THEOREM 4 : If each T_n is an operator, $\| T_n \|$ is uniformly bounded, and $T_n(x)$ converges for an everywhere dense set of points x. Then $T_n(x)$ converges for each x.

3. Hahn-Banach Extension Theorem.

DEFINITION: $M_1 \dot{+} M_2$, $M_1 \oplus M_2$, etc.

LEMMA 1 : (a) If M is a closed linear manifold and x_1 is not in M, then $d(x_1, M) = \inf_{x \in M} \| x - x_1 \| > 0$ (Complement of M is open).

(b) If $M_1 \cap M_2 = \{\Theta\}$, then each element in $M_1 \dot{+} M_2$ has a unique representation in the form $x_1 + x_2$ (obvious).

(c) If M_0 is a closed linear manifold, and x_1 is not in M_0, then

$M_1 = M_0 \dot{+} \lambda x_1$ is a closed linear manifold.

LEMMA 2: If f is a bounded linear function on a linear manifold M, there is a unique continuous extension $\bar{f}$ to $\bar{M}$ and $\bar{f}$ is bounded with the same bound as f.

LEMMA 3: Suppose E is a real Banach space; suppose f is a bounded linear function on a closed linear manifold $M_0 (\subset E)$, suppose x_1 is not in M_0, and $M_1 = M_0 \dot{+} \{\lambda x_1\}$. Then $\exists$ a linear function f_1 on M_1 such that

$$f_1(x) = f(x) \text{ for } x \text{ on } M, \quad |f_1(x)| \leq \|f\|_M \cdot \|x\| \text{ for } x \text{ on } M_1.$$

LEMMA 4: Result of Lemma 3 for E a complex Banach space.

DEFINITION: A separable space.

THEOREM 1 (Hahn-Banach Extension Theorem): Suppose E is a real or complex Banach space, M is a linear manifold in E, f is a linear function defined on M with $|f(x)| \leq M \cdot \|x\|$ for all $x \in M$. Then $\exists$ a linear functional $F(x)$ on $E \ni F(x) = f(x)$ on M and $|F(x)| \leq M \|x\|$ for all x on E.

First proof for a separable space E. Remarks about non-separable spaces, transfinite processes, etc. Examples of non-separable spaces.

THEOREM 2: For any x_0 in E_0, $\exists$ a linear functional f such that

$$f(x_0) = \|x_0\| \text{ and } \|f\| = 1.$$

4. Linear functionals on certain spaces.

THEOREM 1: (a) If f is a linear functional on E_N, then

$$f(x) = \sum_{i=1}^{N} a_i x^i \quad \text{and} \quad \|f\| = \left[\sum_{i=1}^{N} |a_i|^2\right]^{1/2}$$

(b) If f is a linear functional on m_N, then $f(x) = \sum a_i x^i$, $\|f\| = \sum |a_i|$.

THEOREM 2: If f is a linear functional on C_S, $S = [a,b]$, $\exists$ a $y(t)$ of bounded variation

$$f(x) = \int_a^b x(t)dy(t) . \quad \|f\| = V_a^b(y) .$$

LEMMA 1: If μ is a measure over set S, S is measurable μ and $S = \bigcup_{n=1}^{\infty} S_n$ where each S_n is measurable μ and $\mu(S_n) < \infty$, then the finite step functions are everywhere dense in $L_p(S,\mu)$ for each $p \geq 1$.

THEOREM 3: If S is as in Lemma 1 and $p > 1$ and f is a linear functional on L_p, then $\exists$ a $y \in L_q$, where $q = \frac{p}{p-1}$ ($\frac{1}{q} + \frac{1}{p} = 1$), such that

$$f(x) = \int_S x(t)y(t)d\mu(t) , \quad \|f\| = \left(\int_S |y(t)|^q d\mu\right)^{1/q}$$

for any representatives $x(t)$, $y(t)$ of x and y.

PROBLEM 1: Find the forms of the most general linear functionals on (a) ℓ_1, (b) ℓ_p, (c) m_0, specifying the norm in each case. Here m_0 is the subspace of elements $(x_1, x_2, \ldots.)$ of $m \ni \lim_{n\to\infty} x_n = 0$.

PROBLEM 2: If S is as in Lemma 1, f is a linear functional on $L_1(S,\mu)$, then $\exists$ a $y(t)$ which is bounded and measurable on S such that

$$f(x) = \int_S x(t)y(t)d\mu \text{ ; also } \|f\| = \operatorname*{ess\,sup}_{t\in S} |y(t)| .$$

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5. Miscellaneous.

THEOREM 1: Suppose G is a closed linear manifold which is a proper subset of a linear manifold D in a Banach space B. Then, for each $\epsilon > 0$, $\exists$ on $x_0 \in D \ni \|x_0\| = 1$ and $\|x_0 - x\| \geq 1 - \epsilon$ for all x in G.

DEFINITIONS: A set of vectors spans a linear manifold M. M is of finite dimensionality. The dimension of M. Linearly independent set. Minimal set of vectors spanning a linear manifold of finite dimension.

THEOREM 2: Suppose M is a finite dimensional linear manifold in a Banach space E. Then

(a) Any minimal set for M is linearly independent.

(b) If 2 linearly independent sets span M, then each vector of each set can be expressed in terms of the vectors of the other set.

(c) If $x_1, \ldots, x_k$ are linearly independent and span M, then

$$\exists \text{ a } c > 0 \ni c\left(\sum |\lambda_i|^2\right)^{1/2} \leq \left\|\sum \lambda_i x_i\right\| \leq \sum_{i=1}^{k} |\lambda_i| \cdot \|x_i\| .$$

(d) M is closed.

THEOREM 3: If M is not finite dimensional, the set of x in M where $\|x\| \leq 1$ is not (sequentially) compact.

6. Weak convergence of elements in a Banach space.

DEFINITION: Conjugate space B^* of linear functionals.

THEOREM 1: If $x \in B$ and $F(f) = f(x)$ for every $f \in B^*$, then F is a linear functional on B^* and $\| F \| = \| x \|_B$.

DEFINITION: Reflexive space.

THEOREM 2: Suppose $\{x_n\}$ is a sequence of elements $\ni |f(x_n)| \leq M_f$ for every n and every f. Then $\| x_n \|$ are uniformly bounded.

PROBLEM 3: Prove Theorem 2. Hint: Use the method of proof of the theorem on the limit of operators, the space being B^*.

DEFINITION: $x_n \to x$ weakly in B.

DEFINITION: "Weak neighborhoods", convergence in a topological space.

THEOREM 3: (a) If $x_n \to x$ then $x_n \to x$

(b) If $x_n \to x$ any subsequence $x_{k_n} \to x$

(c) If $x_n \to x$ and $x_n \to y$, then $x = y$

(d) Suppose $\{x_n\}$ and x_0 are such that each subsequence of $\{x_n\}$ contains a further subsequence which $\to x_0$. Then $x_n \to x_0$.

(e) If $x_n \to x_0$, then $\| x_n \|$ uniformly bounded.

REMARK: A Banach space is a "Hausdorff space" with respect to weak topology.

DEFINITION: Weak Cauchy sequence. Complete with respect to weak convergence.

THEOREM 4: If B is reflexive, then B is complete with respect to weak convergence.

THEOREM 5: A necessary and sufficient condition that $x_n \to x_0$ is that

$\|x_n\|$ uniformly bounded and $f(x_n) \longrightarrow f(x_0)$ for an everywhere dense set of f.

COROLLARY 1: If B^* is separable, any bounded sequence contains a weak Cauchy sequence.

COROLLARY 2: If, also, B is reflexive, each bounded sequence contains a subsequence which converges weakly to some element in B.

THEOREM 6: (a) $x_n \rightharpoonup x$ in $L_p(S,\mu)$ $(p > 1)$, $\longleftrightarrow$ $\|x_n\|$ is uniformly bounded and $\int_e x_n(t)d\mu \longrightarrow \int_e x(t)d\mu$ for each measurable e with $\mu(e) < \infty$.

(b) If S is R_N and μ is Lebesgue measure $x_n \rightharpoonup x$ in $L_p(S,\mu)$ $\longleftrightarrow$ the conditions in (a) hold where e is any cell with rational vertices.

THEOREM 7 (Remark): If $S = [a,b]$, $x_n \rightharpoonup x$ in $C(S)$ $\longleftrightarrow$ $\|x_n\|$ are uniformly bounded and $x_n(t) \longrightarrow x(t)$ for each t.

THEOREM 8: (a) $L_p(R_N,m)$ is reflexive and separable for each $p > 1$.

(b) The same is true if R_N is replaced by a domain $G \subset R_N$.

PROBLEM 4: Prove Theorem 8.

THEOREM 9: Closed linear manifolds are also closed with respect to weak convergence; i.e. if $x_n \rightharpoonup x_0$, $\exists$ a sequence $y_n = \sum_{i=1}^{n} a_{ni}x_i$ $\ni$ $y_n \longrightarrow x_0$.

7. Completely continuous operators; eigenvalues; the Riesz theory.

DEFINITION: Completely continuous operators.

THEOREM 1: If T_n and T are operators from B to B,

each T_n is completely continuous and $\| T_n - T \| \longrightarrow 0$, then T is completely continuous.

THEOREM 2: If T_1 is an operator from B to B_1 and T_2 is one from B_1 to B_2, and one of T_1 or T_2 is completely continuous, then $T_2 T_1$ is completely continuous.

THEOREM 3: If $T_1, \ldots, T_n$ are completely continuous from B to B_1 and $T = \sum c_i T_i$, then T is completely continuous from B to B_1.

From now on, confine ourselves to operators from B to B.

EXAMPLES: 1. $B = C(S)$, $S = [a,b]$. $Tx = y$, $y(s) = \int_a^b K(s,t)x(t)dt$

K, continuous for $a \leq s \leq b$, $a \leq t \leq b$.

2. Same B, same notation, $y(s) = \int_a^b \dfrac{K(s,t)x(t)dt}{|s-t|^\alpha}$, $0 < \alpha < 1$.

Proof: Define $T_n x = y_n$ defined by

$$y_n(s) = \int_a^b K_n(s,t)x(t)dt, \quad K_n(s,t) = \begin{cases} \dfrac{K(s,t)}{|s-t|^\alpha} \text{ if } |s-t| \geq \dfrac{1}{n} \\ n^\alpha K(s,t) \text{ if } |s-t| \leq \dfrac{1}{n}. \end{cases}$$

$$|y_n(s) - y(s)| \leq \int_{\sigma_n(s)} \frac{1}{|s-t|^\alpha} - n^\alpha \cdot |K(s,t)| \cdot |x(t)| dt$$

$$\leq M \cdot \| x \| \cdot \frac{2}{n^{1-\alpha}} \qquad \| T_n - T \| \leq \frac{2K}{n^{1-\alpha}}$$

3. Same as in 1 with $B = L_2(S)$, $S = [a,b]$.

4. Same formula as in 1 but with $K(s,t)$ in L_2 on square.

THEOREM 4: If T is completely continuous, the manifold M of $x \ni$

$x - Tx = \Theta$ is a closed linear manifold of finite dimension. Also $\exists$ an $m > 0$

$$\| x - Tx \| \geq m d(x,M) .$$

THEOREM 5: If T is completely continuous and $x - Tx = \Theta \longleftrightarrow x = \Theta$, then I-T has a bounded inverse.

Consider $I - \lambda T$, i.e. $x - \lambda Tx$ from now on.

DEFINITIONS: Eigenvalue. Spectrum. Complex B.

Lemma: $\| T_1 \cdots T_n \| \leq \| T_1 \| \cdots \| T_n \|$. By induction.

THEOREM 6: If T is any operator $I - \lambda T$ has an inverse for each λ with $|\lambda| < \dfrac{1}{\| T \|}$;

THEOREM 7: If $\lambda_1, \ldots, \lambda_n$ are all distinct and non-zero and $x_i = \lambda_i Tx_i$, $i = 1, \ldots, n$, then the $x_1, \ldots, x_n$ are linearly independent.

THEOREM 8: If T is a completely continuous operator, its eigenvalues are isolated.

8. Hilbert space; unitary space.

DEFINITION: An N-dimensional unitary space (real or complex).

DEFINITION: Hilbert space (real or complex) $\mathcal{H}$.

THEOREM 1: (a) $(x, \sum_{i=1}^{n} d_i y_i) = \sum_{i=1}^{n} \bar{d}_i (x, y_i)$

(b) $|(x,y)| \leq \| x \| \cdot \| y \|$ (Schwarz)

DEFINITION: x orthogonal to y; x orthogonal to M.

DEFINITION: n.o. set. Complete n.o. set.

LEMMA: If M is a linear manifold of finite dimensionality and x is not in M, $\exists$ a y_0 in M nearest x and $y_0 - x$ is orthogonal to M.

THEOREM 2: If $\mathcal{H}$ is separable, $\mathcal{H}$ contains a complete n.o. set.

THEOREM 3: If $\{e_i\}$ is a complete n.o. set, and $x \in \mathcal{H}$, then the series $\sum_{n=1}^{\infty} |(x,e_n)|^2 < \infty$ and if we define $x_N = \sum_{n=1}^{N} (x,e_n)e_n$, then $\|x_N - x\| \longrightarrow 0$. Moreover if $(x^1, x^2, \dots) \in \ell_2$ and $x_N = \sum_{n=1}^{N} x^n e_n$, then $\exists$ on $x \ni \|x_N - x\| \longrightarrow 0$.

THEOREM 4: If $\{e_i\}$ is a complete n.o. set, $x = \sum x^i e_i$, $y = \sum y^i e_i$, then $(x,y) = \sum x^i \bar{y}^i$.

DEFINITION: Conjugate-linear functional.

LEMMA: f conjugate-linear $\Longleftrightarrow$ $\bar{f}$ is linear.

THEOREM 5: If $f \in \mathcal{H}^*$ $\exists$ a $y \in \mathcal{H}$ $\ni$ $f(x) = (x,y)$ for all x. If f is conjugate linear, there is a $y \in \mathcal{H}$ $\ni$ $f(x) = (y,x)$ for all x.

THEOREM 6: If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{H}$ is reflexive.

THEOREM 7: If M is any closed linear manifold and $x_0 \in \mathcal{H} - M$, $\exists$ a y_0 in M which is nearest x_0 and $(x_0 - y_0)$ is orthogonal to M.

DEFINITION: Projection of an element on M; a projection operator.

CHAPTER I

SEMI-CLASSICAL RESULTS

1.1. Notations. In this course, we shall use the following notations and terminology: If G is a domain, ∂G denotes its boundary and $\bar{G}$ denotes its closure; all domains will be bounded unless otherwise specified. If D is a domain with $\bar{D}$ compact and $\bar{D} \subset G$, we write $D \subset\subset G$. If G is a domain and $h > 0$, G_h denotes the set of all points x in $G \ni B(x,h) \subset G$. If φ is a function, $\lambda(\varphi)$ denotes its support, which is the closure of the set where φ does not vanish.

A function $f \in C^m(G)$ iff (if and only if) f is continuous together with its partial derivatives of order $\leq m$ on G; $f \in C^m_\mu(G)$ for $0 \leq \mu \leq 1$ iff $f \in C^m(G)$ and its derivatives of order $\leq m$ satisfy a uniform Hölder condition with exponent μ (Lipschitz if $\mu = 1$) on each compact subset of G; $f \in C^m(\bar{G})$ iff $f \in C^m(G)$ and f and its derivatives of order $\leq m$ can be extended to $\bar{G}$ and hence to a domain $\supset \bar{G}$; $f \in C^m_\mu(\bar{G})$ iff f can be extended to $\in C^m_\mu(\Gamma)$ for some $\Gamma \supset G$. Similar definitions apply for $f \in C^\infty(G)$, $C^\infty(\bar{G})$, or for f to be analytic on G or $\bar{G}$. A mapping $x^\alpha = x^\alpha(y^1,\ldots,y^\nu)$, $\alpha = 1,\ldots,\nu$ from a domain $\tilde{G}$ onto G is of class C^m iff it is 1-1 and each of the functions x^α and y^ρ of the inverse $\in C^m(G)$ or $C^m(\bar{G})$; the mapping is also regular iff all the derivatives of order $\leq m$ are uniformly bounded; mappings of class C^∞ or analytic are defined similarly. The class $C^m_c(G)$ (or $C^m_{\mu c}(G)$ or $C^\infty_c(G)$, etc.) consists of all $f \in C^m(G)$ (or etc.) with compact support. A domain G is of class C^m (C^m_μ, C^∞, analytic) iff it is bounded and each point P_0 of ∂G is in a domain η which is the image under a 1-1 mapping τ of class C^m (C^m_μ, C^∞ or analytic) of a sphere $B(0,R)$ in which σ_R (see just below) is carried into $\eta \cap \partial G$ and G_R is carried into $\eta \cap G$.

We say $f \in L_p(G)$ iff f is measurable and $|f|^p$ is summable on G; but we also use $L_p(G)$ to denote the space of classes of equivalent functions with norm

$$\|f\|^{\rho}_{p,G} = \left\{ \int_G |\bar{f}(x)|^p dx \right\}^{1/p}$$

for any representative $\bar{f}$ of the element f. If $f \in \overset{\circ}{C}_{\mu}(\mathcal{S})$, $h_{\mu}(f)$ denotes its μ-Hölder constant, i.e.

$$\sup_{\substack{x_1, x_2 \in \bar{G} \\ x_1 \neq x_2}} |x_2 - x_1|^{-\mu} \cdot |f(x_2) - f(x_1)| \qquad (x_k = (x_k^1, \ldots, x_k), k=1,2)$$

If v is a vector or tensor (such as $v_{\gamma}^{\alpha\beta}$), $|v|$ denotes its length, i.e. the square root of the sum of the square of its components. If S is a measurable set, $|S|$ denotes its measure. $B(x_o,R)$ denotes the open sphere with center at x_o and radius R. E_{ν} denotes Euclidean ν-space (metric included); E_{ν}^{+} is that part of E_{ν} where $x^{\nu} > 0$, E_{ν}^{-} that where $x^{\nu} < 0$, $\sum = \partial B(0,1)$.

We have many occasions to work in the spheres $B(0,R)$; we often abbreviate $B(0,R)$ (and often $B(x_o,R)$ to B_R. We denote the hemisphere $G_R = B_R \cap E_{\nu}^{+}$, the hemisphere $G_R^{-} = B_R \cap E_{\nu}^{!}$. Also $\sigma_R = \bar{G}_R \cap \bar{G}_R^{-} \cap B_R$, $\sum_R = \partial B_R \cap E_{\nu}^{+}$, $\sum_R^{-} = \partial B_R \cap E_{\nu}^{-}$.

If $u \in C'(G)$, $u_{,\alpha}$ denotes $\partial u / \partial x_j^{\alpha}$ similar notations hold for higher derivatives. In boundary integrals, $dx_{\alpha}^{!}$ stands for $(n \cdot e_{\alpha})dS$ where n is the unit outer normal, e_{α} is the unit vector in the x^{α} direction, and dS is the $(\nu-1)$-surface element; thus Green's theorem (D of class C', $u \in C'(G)$) reads

$$\int_{\partial D} u \, dx_{\alpha}^{!} = \int_D u_{,\alpha} \, dx$$

$$\Gamma_{\nu} = |\partial B(0.1)|, \qquad \gamma_{\nu} = |B(0,1)| \text{ in } \nu\text{-space.}$$

If $u \in C^m(G)$, $\alpha = (\alpha_1, \ldots, \alpha_{\nu})$ where each α_{β} is an integer ≥ 0, $|\alpha|$ denotes

$\alpha_1 + \dots + \alpha_\nu$, and $D^\alpha u$ denotes $\partial^{|\alpha|} u . \partial x_1^{\alpha_1}, \dots, \partial x_\nu^{\alpha_\nu}$; $\nabla^k u$ denotes all the derivatives $D^\alpha u$ with $|\alpha| = k$ and $|\nabla^k u|^2 = \sum_{|\alpha|=k} |D^\alpha u|^2$.

The cell $a^\alpha \le x^\alpha \le b^\alpha$, $\alpha = 1, \dots, \nu$, is denoted by $[a,b]$. There are times when we wish to study the behavior of a function as a function of some one variable x^α or with respect to all variables except x^α. When this is the case, we write $x = (x^\alpha, x'_\alpha)$ and $f(x) = f(x^\alpha, x'_\alpha)$ and denote $(\nu-1)$ dimensional integrals over cells $[a'_\alpha, b'_\alpha]$ on x^α = constant by $\int_{a'_\alpha}^{b'_\alpha} f(x^\alpha, x'_\alpha) dx'_\alpha$, etc.

1.2. Elementary properties of harmonic functions.

DEFINITION 1.2.1 : u is harmonic on $G \Longleftrightarrow u \in C^2(G)$ and

$$\Delta u(x) = \sum_{\alpha=1}^{\nu} u_{,\alpha\alpha}(x) = 0 , \qquad x \in G.$$

THEOREM 1.2.1: If u is harmonic on G, $D \subset\subset G$ and D is of class C' and $\overline{B(x_0,R)} \subset G$, then

(1.2.1) $\quad \int_{\partial D} u_{,\alpha}\, dx'_\alpha = 0$

(1.2.2) $\quad u(x_0) = \Gamma_\nu^{-1} \int_{\partial B(x,R)} u(x)\, d\Sigma$

(1.2.3) $\quad u(x_0) = |B(x_0,R|^{-1} \int_{B(x_0,R)} u(x) dx$

Proof: The first follows from Green's theorem. If $D = B(x_0, r)$, $0 < r \le R$, (1.2.1) takes the form

(1.2.4) $\quad \int_{\partial B(x_0,r)} u_r dS = 0.$

where u_r denotes the radial derivative. Let (r,p) be polar coordinates with pole at x_0, $r = |x - x_0|$ and p on $\Sigma = \partial B(0,1)$. Then (1.2.4) is equivalent to

(1.2.5) $\quad \int_\Sigma v_r(r,p)\, d\Sigma = 0, \quad v(r,p) = u(x_0 + r\zeta_p), \quad \zeta_p = \overrightarrow{0p}$.

Integrating (1.2.5) from 0 to R yields (1.2.2). Using (1.2.2) with R replaced by r, we obtain (1.2.3) by multiplying both sides of (1.2.2) by $\Gamma_\nu r^{\nu-1}$ and integrating wrt. r from 0 to R.

THEOREM 1.2.2: Suppose u is harmonic on the domain G, $x_0 \in G$, u takes on its maximum value at x_0. Then $u(x) \equiv u(x_0)$.

Proof: Use the mean value theorem and connectedness of G.

THEOREM 1.2.3: If u is continuous and (1.2.3) holds for every $B(x_0,R) \subset\subset G$, then u is harmonic. A harmonic function has derivatives of all orders which are harmonic.

Proof: Since (1.2.3) holds, we see that $u \in C'(G^0)$ with

$$u_{,\alpha}(x) = |B(x_0,R)|^{-1} \int_{\partial B(x_0,R)} u(x)\, dx'_\alpha \quad \text{if } B(x_0,R) \subset\subset G,\ \alpha = 1,\dots,\nu$$

But then by Green's theorem, we see that $u_{,\alpha}$ also satisfies (1.2.3). By induction, we see that all derivatives are continuous and satisfy (1.2.3).

So, suppose $x_0 \in G$. Expanding in Taylor's series, we obtain

$$(1.2.6) \quad u(x) - u(x_0) = (x^\alpha - x_0^\alpha) u_{,\alpha}(x_0) + \tfrac{1}{2!}(x^\alpha - x_0^\alpha)(x^\beta - x_0^\beta) u_{,\alpha\beta}(x_0) + R(x_0,x)$$

$$|R(x,x_0)| \leq M \cdot |x - x_0|^3 \text{ if } |x - x_0| \leq R, \quad B(x_0,R) \subset\subset G.$$

It follows, by integrating (1.2.6) over $B(x_0,r)$, dividing by $r^2 |B(x_0,r)|$, using (1.2.3), and letting $r \to 0$, that $\Delta u(x_0) = 0$.

3. EXERCISES

1. Show that if u is harmonic and in $L_p(G)$, then

$$|u(x)|^p \leq |B(x,r)|^{-1} \int_{B(x,r)} |u(y)|^p dy \quad \text{if } B(x,r) \subset G$$

2. If u is harmonic and $u \in L_2(G)$, then $|\nabla_2 u(x)| \leq C(\nu) \|u\|_2^0\, \delta_x^{-1-\nu/2}$

3. Show that if u is harmonic on G, $|u(x)| \leq M$ there, δ_x denotes the distance of x from ∂G for $x \in G$, there exists a constant C, depending only on ν, such that

$$|\nabla^k u(x)| \leq k!\, e^{k-1} C^k M \delta_x^{-k} \qquad , \qquad k \geq 1$$

1.3. Poisson's integral formula; elementary functions; Green's functions.

Suppose G is a bounded domain of class C' and u and v are of class C'' on $\bar{G} = G \cup \partial G$. Then, from Green's theorem, we obtain the formula

$$(1.3.1) \qquad \int_G (u\Delta v - v\Delta u)dx = \int_{\partial G} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS$$

$$= \int_{\partial G} (uv_{,\alpha} - vu_{,\alpha}) dx'_\alpha$$

n being the exterior normal.

Next, it is a well-known and easily verified fact that

$$f(y) = \begin{cases} |y|^{2-\nu}, & \text{if } \nu > 2 \\ \log |y| & \text{if } \nu = 2 \end{cases}$$

is harmonic if $y \neq 0$. Moreover

$$\int_{\partial B(0,\rho)} \frac{\partial f}{\partial n} dS = \begin{cases} -(\nu-2) \int_{\partial B(0,\rho)} \rho^{1-\nu} \cdot dS = -(\nu-2)\Gamma_\nu, & \text{if } \nu > 2 \\ 2\pi, & \text{if } \nu = 2 \end{cases}$$

So, let us define

$$(1.3.2) \qquad K_0(y) = \begin{cases} -(\nu-2)^{-1}\Gamma_\nu^{-1} |y|^{2-\nu}, & \text{if } \nu > 2 \\ (2\pi)^{-1} \log |y| & \text{if } \nu = 2 \end{cases}$$

Now, suppose G is bounded and of class C', $u \in C''(\bar{G})$, $\Delta u(x) = f(x)$ on G, and $x_0 \in G$. Suppose $\overline{B(x_0,\rho)} \subset G$ and we apply (1.3.1) to the domain $G - \overline{B(x_0,\rho)}$ with $v(x) = K_0(x-x_0)$. Then we obtain

$$(1.3.3)\qquad \int_{\partial B(x_0,\rho)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = \int_{\partial G}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS + \int_G v(x)f(x)dx$$

Letting $\rho \longrightarrow 0$ in (1.3.3), we obtain

$$(1.3.4)\qquad u(x_0) = \int_{\partial G}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS + \int_G K_0(x-x_0)f(x)dx$$

$$(v(x) = K_0(x-x_0))$$

DEFINITION 1.3.1: The function K_0, defined in (1.3.2) is called the elementary function for Laplace's equation $\Delta u = 0$.

If $f(x) = 0$, then (1.3.4) expresses $u(x_0)$ in terms of its boundary values and those of its normal derivative. However , from the maximum principle, a harmonic function is completely determined by its boundary values alone. If, in (1.3.4), we have $f(x) = 0$ and could take

$$(1.3.5)\qquad v(x) = K_0(x-x_0) + H(x,x_0) \equiv G_0(x_0,x)$$

where H is harmonic in x for each x_0 and so chosen that $v = 0$ on G, then (1.3.4) would express $u(x_0)$ in terms of its boundary values. Such a function v, if it exists, is called a Green's function for G with pole at x_0. By the maximum principle, the Green's function is unique if it exists at all.

From the discussion so far given, it follows that (a) if a Green's function v exists for a given domain G and point x_0 and (b) if u is of class C'' on $\overline{G}$ and harmonic on G, then (1.3.4) expresses $u(x_0)$ in terms of its boundary values. If a Green's function could be found and if it could be shown to be harmonic in x_0 for each x in G, then the function u defined by (1.3.4) with $f = 0$ and u in the boundary integral replaced by a function u^* would be harmonic; it would then remain to show that $u(x) \longrightarrow u^*(x_0)$ as $x \longrightarrow x_0$

for each x_0 on ∂G. And, of course, proving the existence of the Green's function requires proving the existence of harmonic functions having certain given boundary values. This problem is called the Dirichlet problem.

Because of all the problems mentioned in the discussion above, the Dirichlet problem is not usually solved by proving the existence of the Green's function. However, there is one case where this is possible, namely the case when G is a sphere which, obviously, may be assumed to have center at the origin. We now derive the Green's function for such a sphere.

Let $x \in B_R \equiv B(0,R)$ and let x' be the point inverse to x with respect to B_R, that is, the point where

$$x'_\alpha = R^2 x^\alpha / |x|^2$$

Using the spherical symmetry, it is easy to verify that the ratio $|\xi - x|/|\xi - x'|$ is the same for all ξ on ∂B_R so that

$$(1.3.6) \qquad \frac{|\xi - x|}{|\xi - x'|} = \frac{R - |x|}{\frac{R^2}{|x|} - R} = \frac{|x|}{R}$$

Thus we note that if we define

$$(1.3.7) \qquad G(x,\xi) = \begin{cases} \frac{1}{2\pi}\left[\log|\xi - x| - \log|\xi - x'| - \log(|x|/R)\right], & \nu = 2 \\ -(\nu-2)^{-1}\Gamma_\nu^{-1}\left[|\xi - x|^{2-\nu} - |\xi - x'|^{2-\nu}(|x|/R)^{2-\nu}\right] & \nu > 2 \end{cases}$$

then $G(x,\xi)$ is of the form (1.3.5). Moreover, by using the formulas for x' and ξ', we see that

$$(1.3.8) \qquad G(\xi, x) = G(x,\xi)$$

so that G is harmonic in x for each ξ and vanishes for ξ interior to B_R and $x \in \partial B_R$. Thus the function u defined by (1.3.4) with $f = 0$ and $v(\xi) = G(x,\xi)$ is harmonic on B_R. Finally, since the function $u \equiv 1$ and G satisfy the hypotheses of the argument in the paragraph containing equations (1.3.3) and (1.3.4), we conclude that

$$(1.3.9) \qquad \int_{\partial B_R} \frac{\partial G(x,\xi)}{\partial n(\xi)} dS(\xi) \equiv 1 , \qquad x \in B_R$$

By computation from (1.3.7), we see that

$$\frac{\partial G(x,\xi)}{\partial n(\xi)} = R^{-1}\xi^{\alpha} G_{\xi\alpha} = R^{-1}\Gamma_{\nu}^{-1}\xi^{\alpha}[\,|\xi-x|^{-\nu}(\xi^{\alpha}-x^{\alpha}) - (|x|/R)^{2-\nu}|\xi-x'|^{-\nu}(\xi^{\alpha}-x'^{\alpha})]$$

$$= R^{-1}\Gamma_{\nu}^{-1}[\,|\xi-x|^{-\nu}(R^2-\xi\cdot x) - (|x|/R)^{-\nu}|\xi-x'|^{-\nu}(|x|^2-\xi\cdot x)]$$

For ξ on ∂B_R, we may use (1.3.6) to obtain

$$(1.3.10) \qquad \frac{\partial G(x,\xi)}{\partial n(\xi)} = (\Gamma_{\nu}R^{-1})|\xi-x|^{-\nu}(R^2-|x|^2)$$

and thus obtain <u>Poisson's integral formula</u>

$$(1.3.11) \qquad u(x) = (\Gamma_{\nu}R)^{-1}\int_{\partial B_R} |\xi-x|^{-\nu}(R^2-|x|^2)\, u^*(\xi)d\xi$$

for the harmonic function u which takes on given values u^* on ∂B_R.

To see that $u(x) \to u^*(\xi_0)$ as $x \to \xi_0$ if u^* is continuous, $x \in B_R$, $\xi_0 \in \partial B_R$, we note from (1.3.9), (1.3.10), and (1.3.11) that

$$(1.3.12) \qquad u(x) - u^*(\xi_0) = (\Gamma_{\nu}R)^{-1}\int_{\partial B_R} |\xi-x|^{-\nu}(R^2-|x|^2)[u^*(\xi)-u^*(\xi_0)]d\xi$$

To show that this difference $\longrightarrow 0$, we break the integral on the right in (1.3.12) into integrals I_1 over $\partial B_R \cap B(\xi_o, \rho)$ and I_2 over $\partial B_R - B(\xi_o, \rho)$ where we may choose ρ so that $|u^*(\xi) - u^*(\xi_o)| < \varepsilon/2$ for $\xi \in \partial B_R \cap B(\xi_o, \rho)$, ε being given. The reader may complete the proof.

EXERCISE

Complete the proof that $u(x) - u^*(\xi_o) \longrightarrow 0$ in (1.3.12).

<u>1.4 Potentials.</u> In formula (1.3.4) with $v(x) = K_o(x-x_o) = K_o(x_o-x)$, we see that if u is of class C'' on G with

$$\Delta u(x) = f(x) \tag{1.4.1}$$

where G is of class C', then the boundary integrals are harmonic so that the function U defined by

$$U(x) = \int_G K_o(x-\xi) f(\xi)\, d\xi \tag{1.4.2}$$

would differ from u by a harmonic function and hence would also be a solution of (1.4.1).

DEFINITIONS: The equation (1.4.1) is known as <u>Poisson's equation</u> and the function U in (1.4.2) is called the <u>potential of</u> f.

Unfortunately, if f is merely continuous, it does not follow that U is necessarily of class C''; we shall give an example of this later. However, we begin with the following theorem:

THEOREM 1.4.1: <u>If</u> f <u>is bounded and measurable and has compact support</u> $\subset G$<u>(bounded), its potential</u> $U \in C'$

$$U_{,\alpha}(x) = \int_G K_{o,\alpha}(x-\xi) f(\xi)\, d\xi \quad , \qquad M = \sup f(\xi), \tag{1.4.3}$$

(1.4.4) $\quad |U_{,\alpha}(x_2) - U_{,\alpha}(x_1)| \le M[3\rho + \rho\log(3 + \Delta/\rho)], \quad \Delta = \operatorname{diam} G, \ \rho = |x_2 - x_1|$

<u>Proof:</u> We shall first assume that f is continuous everywhere and define $V_\alpha(x)$ to equal the right side of (1.4.3) and define

(1.4.5) $\quad V_{\alpha\rho}(x) = \int_{G-B(x,\rho)} K_{o,\alpha}(x-\xi) f(\xi) d\xi \quad , \qquad x \in E_\nu .$

We note first that the integrals are absolutely convergent and

(1.4.6) $\quad |V_{\alpha\rho}(x) - V_\alpha(x)| \le M \int_{B(x,\rho)} \Gamma_\nu^{-1} |\xi - x|^{1-\nu} d\xi \ = M\rho, \quad M = \sup f(\xi) .$

for all x. Also, if we choose $G' \supset G \cup \overline{B(x,\rho)}$ (remembering that $f(\xi) = 0$ on $E_\nu - G$)

(1.4.7) $\quad V_{\alpha,\beta}(x) = - \int_{\partial B(x,\rho)} K_{o,\alpha}(x-\xi) f(\xi) d\xi'_\beta + \int_{G'-B(x,\rho)} K_{o,\alpha\beta}(x-\xi) f(\xi) d\xi$

$x \in E_\nu .$

By calculating $K_{o,\alpha\beta}$, we see that

(1.4.8) $\quad |K_{,\alpha\beta}(y)| \le \Gamma_\nu^{-1}[\,|y|^{-\nu} + |y|^{-\nu-2}|y^\alpha| \cdot |y^\beta|\,], \ |K_{,\alpha}(y)| = \Gamma_\nu^{-1}|y|^{-\nu}|y^\alpha|$

Thus, if $\overline{B(x,\rho)} \cap G \neq 0$, we may choose $G' = G \cup B(x,2\rho)$ and use (1.4.7) and (1.4.8) to obtain

$$|V_{\alpha,\beta}(x)| \le M + \int_{B(x,\Delta+3\rho)-B(x,\rho)} M\Gamma_\nu^{-1}[\,|\xi-x|^{-\nu} + \nu|\xi-x|^{-\nu-2}|\xi^\alpha - x^\alpha| \cdot |\xi^\beta - x^\beta|\,] d\xi$$

$$\le M\,[1 + C_\nu \int_\rho^{\Delta+3\rho} r^{-1} dr] = M[1 + C_\nu \log(3 + \Delta/\rho)].$$

If $\overline{B(x,\rho)} \cap G = 0$, then the first term in (1.4.7) is 0 and, since $f(\xi) = 0$ outside G, the second integral reduces to an integral over G and

$$\le M\, C_\nu \int_\delta^{\Delta+\delta} r^{-1} dr \le [1 + C_\nu \log(3+\Delta/\rho)] \quad \text{since} \quad \delta \ge \rho \ .$$

Thus (1.4.4) follows with $U_{,\alpha}$ replaced by V_α. Since this inequality does not depend on the modulus of continuity of f, we may approximate any admitted f by continuous functions and (1.4.4) follows in the limit for V_α. That $V_\alpha(x) = U_{,\alpha}(x)$ then follows by integration, using Fubini's theorem.

1.5 General potential theory. In the preceding section, we saw that if f is bounded and measurable, its potential is of class C' and its derivatives are given by formulas of the type

$$(1.5.1) \qquad V(x) = \int_G \Gamma(x-\xi) f(\xi)\, d\xi$$

where Γ satisfies the following general hypotheses:

GENERAL HYPOTHESES ON Γ:

(a) Γ is positively homogeneous of degree $1-\nu$;

(b) $\Gamma(-y) = -\Gamma(y)$ for $y \neq 0$.

(c) Γ is of class C'' on $E_\nu - \{0\}$.

Integrals of the type in (1.5.1) arise in the study of elliptic systems of higher order also, so we devote this section to the study of such integrals and to other useful lemmas.

THEOREM 1.5.1: For any Γ satisfying the general hypotheses,

$$(1.5.2) \qquad \int_{\partial B(x_0,R)} \Gamma(\xi - x)\, d\xi'_\gamma = \int_{\partial B(0,1)} \Gamma(y) y^\gamma dS(\eta) \quad , \; x \in B(x_0, r)$$

Proof: If $x = x_0$, the result is obvious from the homogeneity. Otherwise we set up a correspondence $\xi = \xi(\eta)$ between the points η of $\partial B(0,1)$ and ξ on $\partial B(x_0,R)$ by means of the equation

$$\xi(\eta) = x + \lambda(\eta) \cdot \eta \quad , \quad \lambda(\eta) = |\xi(\eta) - x| > 0$$

Then

$$(1.5.3)\qquad d\,\xi'_\gamma = R^{-1}(\xi^\gamma - x_0^\gamma)\cdot dS(\xi) = R^{-1}(\xi^\gamma - x_0^\gamma)\cdot\lambda^{\nu-1}\sec X(\eta)\,dS(\eta),$$

$$(1.5.4)\qquad \cos X(\eta) = R^{-1}\eta\cdot(\xi - x_0)$$

For a given η, let $x_1(\eta)$ be the foot of the perpendicular from x_0 to the line joining x with $\xi(\eta)$. Then we see that $\xi(\eta)$ is the other intersection of this line with $\partial B(x_0,R)$ and that

$$(1.5.5)\qquad x_1(-\eta) = x_1(\eta), \qquad X(-\eta) = X(\eta),$$

$$(1.5.6)\qquad \xi(\eta) = x_1(\eta) + R\cos X(\eta)\cdot\eta.$$

From the homogeneity and (1,5.6) we have

$$\lambda^{\nu-1}\Gamma(\xi - x) = \Gamma(\eta)$$

$$(1.5.7)\qquad R^{-1}(\xi^\gamma - x_0^\gamma)\sec X(\eta) = R^{-1}(\xi^\gamma - x_1^\gamma)\sec X(\eta) + R^{-1}(x_1^\gamma - x_0^\gamma)\sec X(\eta)$$
$$= \eta^\gamma + R^{-1}[x_1^\gamma(\eta) - x_0^\gamma]\sec X(\eta)$$

Using (1.5.7) and (1.5.3), we see that

$$\int_{\partial B(x_0,R)}\Gamma(\xi - x)\,d\xi'_\gamma = \int_{\partial B(0,1)}\Gamma(\eta)\eta^\gamma dS(\eta)$$
$$+\int_{\partial B(0,1)}\Gamma(\eta)\cdot R^{-1}[x_1^\gamma(\eta) - x_0^\gamma]\sec X(\eta)\,dS(\eta)$$

and the last integral vanishes by virtue of (b) and (1.5.5).

THEOREM 1.5.2: (a) <u>Any two points</u> x_1 <u>and</u> x_2 <u>in</u> B_R <u>can be joined by a path</u> $x = x(s)$, $0 \le s \le \ell$, <u>in</u> B_R <u>such that</u>

$$\int_0^{\ell} (R - |x(s)|)^{\mu-1} ds \leq (2\mu^{-1} + 1) \cdot |x_1 - x_2|^{\mu} \quad , \quad 0 < \mu \leq 1 \; .$$

(b) There is a constant $C(\mu)$ such that any two points of G_R can be joined by a path $x = x(s)$, $0 \leq s \leq \ell$, in G_R such that

$$\int_0^{\ell} \{\delta[x(s)]\}^{\mu-1} ds \leq C(\mu) \cdot |x_1 - x_2|^{\mu} , \qquad 0 < \mu \leq 1 ,$$

where $\delta(x)$ denotes the distance of x from $\partial G_R = \Sigma_R \cup \sigma_R$.

Proof: (a) is easily verified as follows: Let $\rho = |x_1 - x_2|$. If $R \leq \rho < 2R$, choose the polygonal path $x_1 \underline{0} x_2$. If $0 < \rho < R$ and $|x_1| \leq R - \rho$, $|x_2| \leq R - \rho$, choose the segment $x_1 x_2$. If $|x_1| > R - \rho$, $|x_2| \leq R - \rho$, choose the polygonal path $x_1 x_3 x_2$ where x_3 is on $\underline{0} x_1$ with $|x_3| = R - \rho$. If $|x_1| > R - \rho$, $|x_2| > R - \rho$, choose the polygonal path $x_1 x_3 x_4 x_2$ where x_3 is on $\underline{0} x_1$, x_4 is on $\underline{0}\, x_2$, and $|x_3| = |x_4| = R - \rho$.

(b) Since $x_1, \underline{0}$, and x_2 lie in a plane, it is sufficient to prove this for $\nu = 2$; thus C does not depend on ν. It is also sufficient to prove it for $0 < |x_1 - x_2| = \rho = kR$ where $0 < k < 1/3$, say. For such ρ, the set S of x such that $\delta(x) \geq \rho$ is the part of the circle $|x| \leq R - \rho$ where $x^2 (=y) \geq \rho$. For such ρ, we choose the paths $x_1 x_2$ if x_1 and $x_2 \in S_\rho$, $x_1 x_3 x_2$ if $x_1 \notin S_\rho$ and $x_2 \in S_\rho$, and $x_1 x_3 x_4 x_2$ if $x_1 \notin S_\rho$ and $x_2 \notin S_\rho$; here x_3 is the nearest point of S_ρ to x_1 and x_4 is that nearest x_2. A straightforward analysis verifies the result in all cases.

As an immediate consequence of Theorem 1.5.1, we obtain

THEOREM 1.5.3: If Γ satisfies our general hypotheses and is of class

C^n on $E_\nu - \{0\}$, $n \geq 1$, there exist constants $C^{\beta_1 \ldots \beta_n}_{\alpha_1, \ldots \alpha_n}$ which depend on Γ but whose magnitudes depend only on those of $\nabla^p \Gamma$ on $\partial B(0,1)$ for $p \leq n$, such that

$$-\int_{\partial B(x_o,R)} \Gamma_{,\alpha_1 \ldots \alpha_n}(x-\xi)(\xi^{\beta_1} - x^{\beta_1}) \ldots (\xi^{\beta_n} - x^{\beta_n}) d\xi'_\gamma = C^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n \gamma}, x \in B(x_o,R)$$

$$-\int_{\partial B(x_o,R)} \Gamma_{,\alpha_1 \ldots \alpha_n}(x-\xi)(\xi^{\beta_1} - x^{\beta_1}) \ldots (\xi^{\beta_p} - x^{\beta_p}) d\xi'_\gamma = 0, \; p < n, \; x \in B(x_o,R)$$

$$\int_{B(x_o,R) - B(x_1,\rho)} \Gamma_{,\alpha_1 \ldots \alpha_n}(x-\xi)(\xi^{\beta_1} - x^{\beta_1}) \ldots (\xi^{\beta_p} - x^{\beta_p}) d\xi = 0, \; p < n,$$
$$x \in B(x_1,\rho) \subset\subset B(x_o,R) .$$

THEOREM 1.5.4: Suppose Γ satisfies the general hypotheses and $f \in C^0_\mu$ on $\overline{B}_R$ with $0 < \mu < 1$, and suppose U is defined by (1.5.1). Then $U \in C^{1+\mu}$ on $\overline{B}_R$ and

(1.5.8) $$U_{,\alpha}(x) = C_\alpha f(x) + \int_{B_R} \Gamma_{,\alpha}(x-\xi)[f(\xi) - f(x)]d\xi = \lim_{\rho \to 0} \int_{B_R - B(x,\rho)} \Gamma_{,\alpha}(x-\xi) f(\xi) d\xi \; ,$$

$$C_\alpha = + \int_{\partial B(0,1)} \Gamma(y) y^\alpha dS(y) \; , \quad x \in B_R \quad .$$

Moreover, there is a constant C, depending only on μ, ν, and the magnitudes of Γ, $\nabla\Gamma$ and $\nabla^2\Gamma$ on $B(0,1)$, such that

(1.5.9) $$h_\mu(\nabla U) \leq C h_\mu(f) \; , \quad |||\nabla U|||^0 \leq C[|||f|||^0 + \mu^{-1} R^\mu h_\mu(f)]$$

Proof: We begin by extending f to be of class C^0_μ on $\overline{B}_{2R}$ and to have the same bounds $h_\mu(f)$ and $|||f|||^0$. Then

$$U = U_1 - U_2 \; , \qquad U_1(x) = \int_{B_{2R}} \Gamma(x-\xi) f(\xi) d\xi \; ,$$

(1.5.10) $$U_2(x) = \int_{B_{2R}-B_R} (x-\xi)f(\xi)d\xi ,$$

Using Theorem 1.5.3 and the fact that $\Gamma_{,\alpha}(x-\xi) = -\partial\Gamma(x-\xi)/\partial\xi^\alpha$, we obtain, for $x \in B_R$,

(1.5.11) $$U_{2,\alpha}(x) = \int_{B_{2R}-B_R} \Gamma_{,\alpha}(x-\xi)f(\xi)d\xi = \int_{B_{2R}-B_R} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]d\xi$$

(1.5.12) $$U_{2,\alpha\beta}(x) = \int_{B_{2R}-B_R} \Gamma_{,\alpha\beta}(x-\xi)[f(\xi)-f(x)]d\xi$$

From (1.5.12), we obtain

(1.5.13) $$|\nabla^2 U_2(x)| \leq C_1 \cdot h_\mu(f)\cdot(R-|x|)^{\mu-1}$$

Using Theorem 1.5.2 and the fact that

$$|\nabla U_2(x_2)-\nabla U_2(x_1)| \leq \int_0^{\ell} |\nabla^2 U[x(s)]ds|$$

along any path $x = x(s)$, $0 \leq s \leq \ell$, we see that

(1.5.14) $$h_\mu(\nabla U_2) \leq C_2 \cdot h_\mu(f)$$

Now, we define

$$U_{1\rho}(x) = \int_{B_{2R}-B(x,\rho)} \Gamma(x-\xi)f(\xi)d\xi \quad , x \in B_R, \ 0 < \rho < R .$$

Then, for such x and ρ

$$U_{1\rho,\alpha} = U_{3\rho\alpha} + U_{4\rho\alpha}$$

$$U_{3\rho\alpha}(x) = -\int_{\partial B(x,\rho)} \Gamma(x-\xi)[f(\xi)-f(x)]d\xi'_\alpha + C_\alpha f(x)$$

$$U_{4\rho\alpha}(x) = \int_{B_{2R}-B(x,\rho)} \Gamma_{,\alpha}(x-\xi)f(\xi)d\xi = \int_{B_{2R}-B(x,\rho)} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]d\xi$$

Clearly, as $\rho \to 0$, $U_{3\rho\alpha}$ and $U_{4\rho\alpha}$ tend uniformly on $\bar{B}_R$ to

(1.5.15) $$U_{3\alpha}(x) = C_\alpha f(x) \quad , \quad U_{4\alpha} = \int_{B_{2R}} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]d\xi$$

Thus, the formulas (1.5.8) follow and the second inequality in (1.5.9) follows immediately.

To complete the proof of the first inequality in (1.5.9), we note first that

(1.5.16) $$|U_{4\rho\alpha}(x)-U_{4\alpha}(x)| \leq C_3 \cdot h_\mu(f)\rho^\mu \quad , \quad x \in B_{\bar{R}} \quad , \quad 0 < \rho < R .$$

But now, if $x, x_1, x_2 \in B_R$, we obtain

$$U_{4\rho\alpha,\beta}(x) = -\int_{\partial B(x,\rho)} \Gamma_{,\alpha}(x-\xi)[f(\xi)-f(x)]d\xi'_\beta$$

$$+ \int_{B_{2R}-B(x,\rho)} \Gamma_{,\alpha\beta}(x-\xi)[f(\xi)-f(x)]d\xi$$

$$|U_{4\rho\alpha}(x)| \leq C_4 h_\mu(f)\rho^{\mu-1}$$

(1.5.17) $$|U_{4\rho\alpha}(x_2)-U_{4\rho\alpha}(x_1)| \leq C_4 h_\mu(f)\rho^\mu \quad \text{if } |x_2-x_1| = \rho$$

From (1.5.15), (1.5.16), and (1.5.17), we obtain

$$h_\mu(U_{4\alpha}) \leq C_5 h_\mu(f) \quad , \quad h_\mu(U_{3\alpha}) \leq C_6 h_\mu(f), \quad h_\mu(\nabla U_1) \leq C_7 h_\mu(f) .$$

The result follows from this and (1.5.14) .

COROLLARY: <u>Suppose</u> $f \in C^0_\mu(\bar{B}_R)$ <u>and</u> V <u>is its potential. Then</u> $V \in C^2_\mu(\bar{B}_R)$ <u>and</u>

(1.5.18) $$\Delta V(x) = f(x), \quad x \in B_R , \quad h_\mu(\nabla^2 V) \leq C h_\mu(f)$$

Proof: It remains only to show that $\Delta V = f$. From (1.4.3) and (1.5.8), we conclude that

$$V_{,\alpha}(x) = \int_G K_{0,\alpha}(x-\xi)f(\xi)d\xi$$

(1.5.19) $$\Delta V(x) = \sum_{\alpha=1}^{\nu} C_{0\alpha\alpha} f(x) + \int_{B_R} \Delta K_0(x-\xi)[f(\xi)-f(x)]d\xi \quad ,$$

(1.5.20) $$C_{o\alpha\alpha} = + \int_{\partial B(0,1)} K_{0,\alpha}(y) y^{\alpha} dS(y)$$

Since $\Delta K_0(y) = 0$ if $y \neq 0$, the integral in (1.5.19) vanishes. From the formulas (1.3.2) for K_0, we see that

$$\sum_{\alpha=1}^{\nu} C_{o\alpha\alpha} = \int_{\partial B(0,1)} \Gamma_{\nu}^{-1} |y|^{-\nu} \cdot |y|^2 dS(y) = 1 \quad .$$

EXERCISE

1. Prove Theorem 1.5.3.

2. Prove that if f satisfies a Dini condition

$$|f(x_2)-f(x_1)| \leq \varphi(|x_2-x_1|) \quad , \qquad \lim_{\rho \to 0} \varphi(\rho) = 0 \; ,$$

$$\int_0^{2R} \rho^{-1} \varphi(\rho) d\rho < \infty \; , \qquad x_1, x_2 \in \overline{B}_R \quad ,$$

then $U \in C'$ on $\overline{B}_R$ if U is defined by (1.5.1).

1.6. The maximum principle for elliptic equations of the second order.
By combining the results of §§ 1.5 and 1.6, we conclude the following theorem:

THEOREM 1.6.1: Suppose $f \in C_{\mu}^{0}$ on $\overline{B}_R$ and u^* is continuous on ∂B_R. Then there is a unique function u which is continuous on $\overline{B}_R$, coincides with u^* on ∂B_R, $\in C^{2+\mu}(B_r)$ for each $r < R$, and satisfies Poisson's equation

$$\Delta u(x) = f(x) \qquad , \qquad x \in B_R \quad .$$

Proof: For, we may let $u = U + H$ where U is the potential of f and H is that harmonic function which coincides with $u^* - U$ on B_R.

In Chapter III, we shall discuss the existence theory and differentiability properties for the solutions of general elliptic equations of the second order:

$$\text{(1.6.1)} \qquad a^{\alpha\beta}(x)u_{,\alpha\beta}(x) + b^{\alpha}u_{,\alpha} + cu = f$$

We conclude this chapter with a proof due to E. Hopf [ ] of the maximum principle for equations of the form (1.6.1) in which

$$\text{(1.6.2)} \qquad c(x) \equiv 0 \ , \qquad f(x) \geq 0 \ , \qquad x \in G.$$

THEOREM 1.6.2: Suppose $u \in C^2(G)$, $a^{\alpha\beta}$, b^{α}, c, $f \in C^0(G)$, c and f satisfy (1.6.2), u is a solution of (1.6.1), $x_0 \in G$, and u takes on its maximum value at x_0. Then $u(x) = u(x_0)$ for all x on the domain G.

Proof: Suppose $u(x) \not\equiv u(x_0) = M$. Then the set where $u(x) < M$ is open. There is a sphere $B(x_2,R) \subset G$ such that $u(x) < M$ for $x \in \overline{B(x_2,R)} - \{x_1\}$, where $x_1 \in \partial B(x_2,R)$ and $u(x_1) = M$. Finally, there is a sphere $\overline{B(x_1,R)} \subset G$ with $R_1 < R$. Let $S_i = \overline{B(x_2,R)} \cap B(x_1,R)$ and $S_e = \partial B(x_1,R) - \overline{B(x_2,R)}$, so that $S_i \cup S_e = \partial B(x_1,R_1)$. Then

$$u(x) \leq M - \varepsilon \quad \text{on } S_i \quad \text{and } u(x) \leq M \quad \text{on } S_e$$

for some $\varepsilon > 0$.

Now, let

$$h(x) = e^{-\gamma r^2} - e^{-\gamma R^2} \qquad , \qquad r = |x - x_2|$$

Letting $L\varphi$ stand for $a^{\alpha\beta}\varphi_{,\alpha} + b^{\alpha}\varphi_{,\alpha}$, we see that

$$e^{\gamma r^2} Lh = 4\gamma^2 a^{\alpha\beta}(x^\alpha - x_2^\alpha)(x^\beta - x_2^\beta) - 2\gamma[a^{\alpha\beta}\delta_{\alpha\beta} + b^\alpha(x^\alpha - x_2^\alpha)] .$$

We can choose γ so large that $Lh(x) > 0$ in $\overline{B(x_1,R_1)}$. Finally

(1.6.3) $\quad h(x) < 0$ on S_e , $\quad h(x_1) = 0$.

Let

(1.6.4) $\quad v(x) = u(x) + \delta h(x)$, $\quad \delta > 0$,

where δ is small enough so that $v(x) < M$ on S_i. From (1.6.3), we see that $v(x) < M$ on $\partial B(x_1,R_1)$, $v(x_1) = M$ so that v has a maximum at a point x_3 in $B(x_1,R_1)$ while $L(v) > 0$ there.

But this would imply that (since all $v_{,\alpha}(x_3) = 0$)

(1.6.5) $\quad a^{\alpha\beta}(x_3)v_{,\alpha\beta}(x_3) > 0$ but $v_{,\alpha\beta}(x_3)\eta^\alpha\eta^\beta \leq 0$ for all η .

Now, we may define new variables y by the rotation

$$y^\gamma = c_\alpha^\gamma (x^\alpha - x_3^\alpha) , \quad \zeta^\gamma = c_\alpha^\gamma \eta^\alpha$$

where the matrix c is chosen so that $c^{-1}a(x_3)c = \acute{a}(0)$ is diagonal. Then (1.6.5) is equivalent to

(1.6.6) $\quad \sum_{\gamma=1}^{\nu} \acute{a}^{\gamma\gamma}(0)w_{,\gamma\gamma}(0) > 0$ but $w_{,\gamma\delta}(0)\zeta^\gamma\zeta^\delta \leq 0$ for all ζ

where all the $\acute{a}^{\gamma\gamma} > 0$. But the first inequality in (1.6.6) implies that some one $w_{,\gamma\gamma}(0) > 0$ which contradicts the second.

COROLLARY: If u and the coefficients satisfy the conditions of Theorem 1.6.2 except that we require $c(x) \leq 0$, then u cannot have a positive maximum. If, also, $f(x) \equiv 0$, then u has neither a positive maximum nor a negative minimum.

CHAPTER 2

THE SPACES H_p^m AND H_{p0}^m

2.1 The spaces H_p^m and H_{p0}^m. In this section, we define these spaces and prove a few theorems concerning them.

DEFINITION 2.1.1 : Given a function u which is locally summable on a domain G, we define the G-Schwartz distribution L_u corresponding to u by the equation

$$(2.1.1) \qquad L_u(g) = \int_G u(x)g(x)dx \qquad g \in C_c^{\infty}(G)$$

We say that a G-Schwartz distribution $\in L_p$ iff it corresponds in the sense above to a function u in $L_p(G)$.

DEFINITION 2.1.2: Given a G-Schwartz distribution L, we define its partial derivative $D_\alpha L$(α a single index) with respect to x^α by the usual equation

$$(2.1.2) \qquad (D_\alpha L)(g) = L(-g_{,\alpha}) \;;$$

The higher partial derivatives of L are defined by induction or by the equation

$$(2.1.3) \qquad (D_\alpha L)(g) = L[(-1)^{|\alpha|}D_\alpha g] \qquad (\alpha = \alpha_1 \dots \alpha_\nu) .$$

DEFINITION 2.1.3: A function u is of class H_p^m on G iff $u \in L_p(G)$ and all the partial derivatives of order $\leq m$ of its corresponding G-Schwartz distribution $\in L_p$. The function corresponding to the derivative $D_\alpha L_u$, $\alpha = \alpha_1 \dots \alpha_n$ where $0 \leq n \leq m$, is denoted by $D_\alpha u$ or $u_{,\alpha}$ and is called the corresponding distribution derivative of u ; we make the convention that $D_\alpha u = u$ if $|\alpha| = 0$.

REMARKS: It is clear that if u is of class H_p^m on G, its distribution derivatives are determined only up to additive null functions; moreover if u^* differs from u by a null function, then u^* is also of class H_p^m on G and has the same distribution derivatives. The definitions above extend to vector functions.

THEOREM 2.1.1: The space $H_p^m(G)$ of classes of equivalent vector functions of class H_p^m on G with norm defined by

$$(2.1.4) \qquad \|u\|_p^m = \left\{\int_G \Big[\sum_{i=1}^{N} \sum_{0\leq|\alpha|\leq m} |D_\alpha u^i|^2\Big]^{p/2} dx \right\}^{1/p} \qquad (u = u^1, \ldots, u^N)$$

is a Banach space. If $p = 2$, the space is a Hilbert space if we define

$$(2.1.5) \qquad (u,v)_2^m = \int_G \sum_{i=1}^{N} \sum_{0\leq|\alpha|\leq m} D_\alpha u^i \overline{D_\alpha v^i}\, dx .$$

Proof: The only property requiring proof is the completeness. So suppose $\{u_n\}$ is a Cauchy sequence in H_p^m. Then each of the sequences $D_\alpha u_n$, with $0\leq|\alpha|\leq m$, is a Cauchy sequence in $L_p(G)$ and so converges in $L_p(G)$ to some vector function φ_α ($= u$ if $|\alpha| = 0$). But, for each α with $0\leq|\alpha|\leq m$, it follows that

$$\lim_{n\to\infty} D_\alpha L_{u_n}(g) = D_\alpha L_u(g) \qquad , \qquad g \in C_c^\infty(G)$$

so that the φ_α are the corresponding distribution derivatives of u.

REMARKS: It will be convenient to call these elements of H_p^m functions and to say that u is continuous, harmonic, etc., iff some representative of the class forming the element has these properties.

THEOREM 2.1.2: (a) If $u \in H_p^m(G)$ and $D \subset G$, then $u \in H_p^m(D)$.

(b) If $u \in C_1^{m-1}$ (i.e. $u \in C^m$ with all the $D_\alpha u$ Lipschitz with $0 \leq |\alpha| \leq m-1$) on each domain $D \subset\subset G$ and if u and its partial derivatives of order $\leq m$ all $\in L_p(G)$ then $u \in H_p^m(G)$.

(c) If $u \in H_p^m$ on each domain $D \subset\subset G$ and if all the $D_\alpha u$, with $0 \leq |\alpha| \leq m$, $\in L_p(G)$, then $u \in H_p^m(G)$.

(d) If $u \in H_p^m(G)$, $g \in C_1^{m-1}$ on G, and g has compact support on G, then

(2.1.6) $$\int_G (-1)^{|\alpha|} g_{,\alpha}(x)u(x)dx = \int_G g(x)u_{,\alpha}(x)dx, \qquad 0 \leq |\alpha| \leq m$$

For (a), (b), and (c) are obvious and (d) follows from the definition of distribution derivative (def. 2.1.2) by a straightforward approximation (of the function g).

DEFINITION 2.1.4: A function φ is a (Friedrichs) mollifier iff $\Lambda(\varphi) \subset B(0,1)$, $\varphi \in C^\infty$ everywhere and

(2.1.7) $$\int_{B(0,1)} \varphi(x)dx = \int_E \varphi(x)dx = 1 .$$

Suppose φ is a given mollifier, u is locally summable on G, and $\rho > 0$. We define the φ-mollified function u_ρ by

(2.1.8) $$u_\rho(x) = \rho^{-\nu} \int_G \varphi[\rho^{-1}(\xi - x)]u(\xi)d\xi, \qquad x \in G_\rho$$

LEMMA 2.1.1: Suppose φ is any mollifier, $u \in L_p(G)$, and u_ρ is defined by (2.1.8). Then $u_\rho(x) \to u(x)$ almost everywhere and $u_\rho \to u$ in L_p on each $D \subset\subset G$ as $\rho \to 0$. For each $\rho > 0$, $u_\rho \in C^\infty(G_\rho)$ and its partial derivatives are obtained by differentiating under the integral sign.

Proof: For almost every x,

$$\lim_{\rho \to 0} |B(x,\rho)|^{-1} \int_{B(x,\rho)} |u(\xi) - u(x)| d\xi = 0.$$

For any such x,

(2.1.9) $$|u_\rho(x)-u(x)| = \left|\rho^{-\nu} \int_{B(x,\rho)} \varphi[\rho^{-1}(\xi-x)]\cdot[u(\xi)-u(x)]d\xi\right| \longrightarrow 0.$$

Next, suppose $D \subset\subset G$. Then we may find a domain $\Delta \ni D \subset\subset \Delta \subset\subset G$ and a sequence $\{v_n\}$ of continuous functions on $\bar{\Delta}$ such that $v_n \longrightarrow u$ in $L_p(\bar{\Delta})$. Now, from (2.1.8), we obtain

(2.1.10) $$|v_{n\rho}(x) - u_\rho(x)| \leq K\rho^{-\nu} \int_{B(x,\rho)} |v_n(\xi) - u(\xi)| d\xi \quad , \quad x \in D,\ B(x,\rho) \subset \Delta.$$

Applying the Hölder inequality to the right side of (2.1.10) and then integrating over D, assuming $\bar{D} \subset \Delta_\rho$, we obtain

(2.1.11) $$\int_D |v_{n\rho}(x)-u_\rho(x)|^p dx \leq K' \int_\Delta |v_n(\xi)-u(\xi)|^p d\xi$$

Hence, if $\bar{D} \subset \Delta_{\rho_0}$, and $\epsilon > 0$, we first choose an n so large that $\|v_n-u\|_p^0 < \epsilon/3$ and $\|v_{n\rho}-u_\rho\|_p^0 < \epsilon/3$, the norm being that in $L_p(D)$. For that n we can choose a $\delta > 0$ so small that $\|v_{n\rho}-v_n\|_p^0 < \epsilon/3$ for all ρ with $0 < \rho < \delta$ (since $v_{n\rho}$ tends uniformly to v_n on $\bar{D}$). For such ρ, $\|u_\rho - u\|_p^0 < \epsilon$. The last statement follows from the convergence theorems for the Lebesgue integral.

THEOREM 2.1.3: Suppose $u \in H_p^m(G)$ and u_ρ is defined by (2.1.8), φ being a given mollifier. Then $u_\rho \longrightarrow u$ in $H_p^m(D)$ for each $D \subset\subset G$ and

(2.1.12) $$\varphi_{\alpha\rho}(x) = D_\alpha u_\rho(x) \quad \text{if } \varphi_\alpha = u_{,\alpha} \quad , \qquad 1 \leq |\alpha| \leq m.$$

Proof: If $\overline{B(x,\rho)} \subset G$, the function $g_{x\rho}$ defined on G by

$$g_{x\rho}(\xi) = \rho^{-\nu}\varphi[\rho^{-1}(\xi-x)]$$

$\varepsilon C_c^\infty(G)$. Hence, using Lemma 2.1.1 and the definitions, we see that

$$D_\alpha u_\rho(x) = \int_G (-1)^{|\alpha|} D_\alpha(\xi) g_{x\rho}(\xi) u(\xi) d\xi$$

$$= \int_G g_{x\rho}(\xi) D_\alpha u(\xi) d\xi = \psi_{\alpha\rho}(x) \quad .$$

The remaining statements follow from Lemma 2.1.1.

REMARK: It is not known (and the writer believes it is not true) that a function $u \varepsilon H_p^m$ on a domain G with sufficiently wild boundary can be approximated over the whole of G by functions of class C^m on a domain containing $G \cup \partial G$. We shall prove this, however, for domains G of class C^m in Section 2-4.

The following theorem is an immediate consequence of Theorems 2.1.1, 2.1.2 b and c, and 2.1.3 and the details of its proof are left to the reader.

THEOREM 2.1.4: (a) <u>If</u> $u \varepsilon H_p^m(G)$ <u>and</u> $\zeta \varepsilon C_1^{m-1}(G)$ <u>and</u> ζ <u>and its derivatives are bounded on</u> G, <u>then</u> $\zeta u \varepsilon H_p^m(G)$ <u>and its derivatives are obtained from those of</u> ζ <u>and</u> u <u>by their usual formulas</u>.

(b) <u>If</u> $x = x(y)$ <u>is a regular</u> (<u>all derivatives bounded</u>) <u>transformation of class</u> C^m <u>from</u> H <u>onto</u> G, $u \in H_p^m(G)$, <u>and</u> $v(y) = u[x(y)]$ <u>for</u> y <u>on</u> H, <u>then</u> $v \varepsilon H_p^m(H)$ <u>and the derivatives of</u> v <u>are related to those of</u> u <u>by the usual laws of the Calculus.</u>

DEFINITION 2.1.5: A function $u \in H_p^m(G)$ is said to $\in H_{p0}^m(G)$ iff $\exists$ a sequence $\{u_n\}$, each $u_n \in C_c^\infty(G)$, such that $u_n \longrightarrow u$ in $H_p^m(G)$.

THEOREM 2.1.5 (Poincare's inequality): <u>If</u> $u \in H_{p0}^m(G)$ <u>and</u> $G \subset B(x_0, R)$, <u>then</u>

$$(2.1.13) \quad \int_G |\nabla^k u(x)|^p dx \leq p^{k-m} R^{(m-k)p} \int_G |\nabla^m u(x)|^p dx \quad , \quad 0 \leq k < m .$$

Proof: We shall prove this for $m=1, k=0$; the general result follows easily by induction, since if $u \varepsilon H_p^m(G)$, $\nabla^k u \varepsilon H_p^{m-k}(G)$. From Definition 2.1.5, it suffices to prove the theorem in the case that $u \varepsilon C^1[B(x_0,R)]$. Taking polar coordinates (r,ζ) (ζ on $\partial B(0,1)$) with pole at x_0 and setting $v(r,\zeta) = u(x_0 + r\zeta)$, we obtain

$$(2.1.14) \qquad \int_\Sigma |v(r,\zeta)|^p d\Sigma(\zeta) = \int_\Sigma |v(r,\zeta) - v(R,\zeta)|^p d\Sigma(\zeta)$$

$$\leq (R-r)^{p-1} \int_r^R \int_\Sigma |v_{,r}(s,\zeta)|^p dr d\Sigma(\zeta)$$

using the Holder inequality. The result (2.1.13) follows for $m=1$, $k=0$ from (2.1.14) and the fact that

$$\int_{B(x_0,R)} |u(x)|^p dx = \int_0^R r^{\nu-1} \int_\Sigma |v(r,\zeta)|^p dr d\Sigma \quad .$$

COROLLARY: The space $H_{p0}^m(G)$ is a closed linear subspace of $H_p^m(G)$ and, if G is bounded, the norm $\|u\|_{p0}^m$ defined by

$$(2.1.15) \qquad \| u \|_{p0}^m = \left\{ \int_G |\nabla^m u(x)|^p dx \right\}^{1/p}$$

is topologically equivalent to the norm $\| u \|_p^m$ for u on $H_{p0}^m(G)$. If $p = 2$, the inner product may be taken on H_{20}^m to be

$$(2.1.1\ ) \qquad (u,v)_{20}^m = \int_G \sum_{i=1}^N \sum_{|\alpha|=m} u_{,\alpha}(x)\overline{v_{,\alpha}(x)}\, dx \qquad (u = u^1,\ldots,u^N, \text{ etc.})$$

EXERCISE

Prove Theorem 2.1.4.

2.2. The Dirichlet problem for Laplace's equation with generalized boundary values. In this section, we interrupt our study of the spaces H_p^m and H_{p0}^m in order to illustrate their use in the existence theory for elliptic differential equations. We also justify Dirichlet's Principle.

LEMMA 2.2.1: *Suppose* $f \varepsilon L_1$ *on the interval* $[a,b]$ *of* R_1 *and suppose that*

$$\int_a^b f(x)g(x)dx = 0 \tag{2.2.1}$$

for all $g \varepsilon C_c^\infty[a,b]$. *Then* $f(x) = 0$ *almost everywhere. If* (2.2.1) *holds only for all* $g \varepsilon C_c^\infty[a,b]$ *for which*

$$\int_a^b g(x)dx = 0, \tag{2.2.2}$$

then $f(x)$ = *a constant almost everywhere.*

Proof: In the first case, it follows by approximations that (2.2.1) holds for all g which are bounded and measurable on $[a,b]$. Setting $g(x)=\operatorname{sgn} f(x)$, (2.2.1) implies that

$$\int_a^b |f(x)|dx = 0.$$

In the second case, let $g_1 \in C_c^\infty[a,b]$ with

$$\int_a^b g_1(x)dx = 1.$$

Then, if g is any function $\varepsilon\ C_c^\infty[a,b]$, we have

$$g(x) = g^*(x) + g_1(x)\int_a^b g(x)dx \quad , \quad \int_a^b g^*(x)dx = 0$$

and, of course, $g^* \varepsilon\ C_c^\infty[a,b]$. The second result follows.

THEOREM 2.2.1 (Dirichlet's Principle): (a) *Suppose* $u \varepsilon H_2^1(G)$ *and*

(2.2.3) $\int_G v_{,\alpha} u_{,\alpha} dx = 0$ (α a single index)

for all $v \in C_c^\infty(G)$. Then, if $u^* \in H_2^1(G)$ and $u^* - u \in H_{20}^1(G)$,

(2.2.4) $D(u^*,G) = D(u,G) + D(u-u^*,G)$, $(D(\varphi,G) = \int_G |\nabla\varphi|^2 dx)$

(b) <u>If</u> $u \in H_2^1(D)$ <u>for each</u> $D \subset\subset G$ <u>and</u> (2.2.3) <u>holds for all</u> $v \in C_c^\infty(G)$, <u>then</u> u <u>is harmonic on</u> G.

<u>Proof</u>: (a) By approximations, (2.2.3) holds for all $v \in H_{20}^1(G)$. Then (2.2.4) follows by taking $v = u^* - u$.

(b) Suppose $\overline{B(x_0,a)} \subset G$, suppose $0 < \varepsilon < a$, suppose $\varphi(r) \in C^\infty$ for $r \geq 0$ with $\varphi(r) = 0$ for $r \geq a$ and $\varphi(r) = \varphi(0)$ if $0 \leq r \leq \varepsilon$, and let

$$v(x) = \varphi(|x-x_0|) \quad , \quad X(r) = \int_\Sigma u(x_0 + r\zeta) d\Sigma(\zeta). \qquad (\Sigma = \partial B(0,1))$$

Then $v \in C_c^\infty(G)$, $X \in L_1[\varepsilon,a]$. Integrating (2.2.3) by parts, we obtain

$$\int_G u \Delta v dx = \int_\varepsilon^a X(r)[\frac{d}{dr}(r^{\nu-1}\varphi')]dr = \int_\varepsilon^a X(r)\Psi(r)dr = 0$$

where Ψ may be any function $\in C_c^\infty[\varepsilon,a]$ for which (2.2.2) holds. It follows from Lemma 2.2.1 that $X(r) =$ a constant almost everywhere on $[\varepsilon,a]$; since ε and a are arbitrary, this holds for almost all r on $[0,\delta(x_0,\partial G)]$. Calling this constant $\Gamma_\nu \cdot \bar{u}(x_0)$, we obtain by integration that

(2.2.5) $\bar{u}(x_0) = |B(x_0,r)|^{-1} \int_{B(x_0,r)} u(x)dx$, $\overline{B(x_0,r)} \subset G$.

By letting $r \longrightarrow 0$, we see that $\bar{u}(x_0) = u(x_0)$ almost everywhere. Then (2.2.5) holds with u replaced by $\bar{u}$, so that $\bar{u}$ is harmonic by Theorem 1.2.3.

THEOREM 2.2.2: <u>Suppose that</u> $u^* \in H_2^1(G)$ <u>and that</u> G <u>is bounded</u>.

Then there is a unique function $u \in H^1_2(G)$ such that u is harmonic in G and $u^* - u \in H^1_{20}(G)$.

Proof: Setting $u = u^* + U$, we see that (2.2.3) holds iff

(2.2.5) $$\int_G \bar{v}_{,\alpha} U_{,\alpha} dx = - \int_G u^*_{,\alpha} \bar{v}_{,\alpha} dx$$

Using the inner product (2.1.15) with $m = 1$ on H^1_{20}, (2.2.5) becomes

(2.2.6) $$(U,v)^1_{20} = \bar{L}(v) \quad , \quad \bar{L}(v) = - \int_G u^*_{,\alpha} \bar{v}_{,\alpha} dx \quad , \quad v \in H^1_{20}$$

where $\bar{L}$ is clearly a conjugate-linear functional. From Hilbert space theory, it follows that there is a unique U in H^1_{20} such that (2.2.6) holds. Then u satisfies (2.2.3) and is harmonic.

2.3. Further theorems concerning the spaces H^m_p and H^m_{p0}.

For functions in H^m_p on arbitrary regions, we have the following theorems:

THEOREM 2.3.1: (a) Suppose $u \in H^m_{p0}(G)$ and $V(x) = u(x)$ for $x \in G$ and $V(x)=0$ elsewhere. Then $V \in H^m_p(E)$ and $V \in H^m_{p0}(D)$ for any open set $D \supset G$. Moreover $D_\alpha V(x) = D_\alpha u(x)$ on G and $D_\alpha V(x) = 0$ for $x \in E - G$ if $0 \leq |\alpha| \leq m$.

(b) Suppose $u \in H^m_p(G)$, $D \subset G$, $v \in H^m_p(D)$, $v - u \in H^m_{p0}(D)$, $U(x) = v(x)$ on D, and $U(x) = u(x)$ on $G-D$. Then $U \in H^m_p(G)$, $U - u \in H^m_{p0}(G)$, and $D_\alpha U(x) = D_\alpha v(x)$ on D and $D_\alpha U(x) = D_\alpha u(x)$ on $G-D$ if $0 \leq |\alpha| \leq m$.

Proof: (a) follows immediately from the definitions and theorems of Section 2.1.1. If we define $V(x) = v(x) - u(x)$ on D and $V(x) = 0$ elsewhere, then the conclusions of part (a) hold with G replaced by D. But then $U(x) = u(x) + V(x)$ on G so the results in (b) follow.

THEOREM 2.3.2: If $p > 1$, the most general linear functional in $H_p^m(G)$ has the form

$$(2.3.1) \qquad f(u) = \int_G \sum_{i=1}^{N} \sum_{0\leq|\alpha|\leq m} A_i^\alpha(x) u^i_{,\alpha}(x)dx$$

where the $A_i^\alpha \in L_q(G)$ where $p^{-1} + q^{-1} = 1$. If $p = 1$, each linear functional has the form (2.3.1) in which the A_i^α are bounded and measurable.

Proof: It is clear that any expression (2.3.1) defines a linear functional on H_p^m. Conversely, let B_p be the space of all tensors $\{\varphi^i_\alpha\}$ where each $\varphi^i_\alpha \in L_p(G)$ with norm

$$\|\varphi\| = \left\{\int_G \left[\sum_{i=1}^{N} \sum_{0\leq|\alpha|\leq m} |\varphi^i_\alpha(x)|^2 \right]^{p/2} dx \right\}^{1/p}$$

Then, from Theorem 2.1.1, it follows that the subspace M of all tensors φ where $\varphi^i_\alpha = u^i_{,\alpha}$ and $u \in H_p^m(G)$ is a closed linear submanifold of B_p. If we define $F_1(\varphi) = f(u)$ for such tensors φ, we have $\|F\| = \|f\|$ and F_1 can be extended (Hahn-Banach Theorem, see § A) to a linear functional F over B_p with the same norm. But any linear functional F on B_p has the form

$$F(\varphi) = \int_G \sum_{i=1}^{N} \sum_{0\leq|\alpha|\leq m} A_i^\alpha(x) \varphi^i_\alpha(x)dx$$

where the A_i^α have the stated properties.

From Theorem 2.3.2, we immediately obtain:

THEOREM 2.3.3: (a) A necessary and sufficient condition that u_n converges weakly to u ($u_n \rightharpoonup u$) in $H_p^m(G)$ is that each component $u^i_{n,\alpha}$ of u_n converges weakly in $L_p(G)$ to $u^i_{,\alpha}$.

(b) If $u_n \rightharpoonup u$ in $H_p^m(G)$ then $u_n \rightharpoonup u$ in $H_p^m(D)$ for $D \subset G$.

(c) *If* $u_n \to u$ *in* $H_p^m(G)$, G *and* H *are bounded*, $x = x(y)$ *is a regular transformation of class* C^m *from* H *onto* G, $v_n(y) = u_n[x(y)]$ *and* $v(y) = u[x(y)]$, *then* $v_n \to v$ in $H_p^m(H)$.

(d) *If* $u_n \to u$ *in* $H_p^m(G)$, $\zeta \in C_1^{m-1}(G)$ *and all the* $D_\alpha \zeta$ *with* $0 \leq |\alpha| \leq m$ *are bounded on* G, *then* $\zeta u_n \to \zeta u$ *in* $H_p^m(G)$.

(e) *If* $p>1$, *bounded sets in* $H_p^m(G)$ *are conditionally compact with respect to weak convergence in* $H_p^m(G)$.

2.4. *Approximations; boundary values; compactness.*

We wish now to prove that we can approximate to a function $u \in H_p^m(G)$ over the whole of G by functions u_n of class C^m on a domain $\supset G \cup \partial G$, in case G is of class C^m (see Def. 1.1.). We shall also introduce boundary values of such functions on such domains.

LEMMA 2.4.1: *Suppose* $f \in L_p(G)$, e *is a unit vector*, $D \subset G$, *all points* $x + he$ *with* $x \in D$ *and* $0<h<h_0$ *are in* G, *and* $f_h(x) = f(x+he)$ *for* x *on* D *and* $0 < h < h_0$. *Then* $f_h \to f$ in $L_p(D)$.

Proof: The proof is like that of Lemma 2.1.1: Extend f to be zero off of G and approximate to it in $L_p(E)$ by continuous functions f_n, each having compact support. Then

$$\| f_{nh} - f_h \|_p^0 = \| f_n - f \|_p^0 \qquad \text{(norms in } L_p(E))$$

$$\| f_h - f \|_p^0 \leq \| f_h - f_{nh} \|_p^0 + \| f_{nh} - f_n \|_p^0 + \| f_n - f \|_p^0 \quad ;$$

the first and last terms may be made $< \varepsilon/3$ by choosing a large n and, for that n, the middle term may be made $< \varepsilon/3$ by choosing h small enough.

THEOREM 2.4.1: Suppose $u \in H_p^m(G)$, G is bounded and of class C^m, D is open,

and $\bar{G} \subset D$. Then there exist a constant $C = C(G,D,m,p)$ and a function $U \in H^m_{p0}(D)$ with support interior to D such that $\| U\|^m_{pD} \leq C\| u \|^m_{pG}$ and $U(x) = u(x)$ for $x \in G$.

Proof: There is a partition of unity on $\bar{G} = G \quad G$ by functions ζ_s, $1 \leq s \leq S$, each of class C^m on $\bar{G}$ with support either interior to G or in a boundary neighborhood $N_s \subset D$, where N_s is the image of $B(0,1)$ under a regular transformation γ_s: $x = x_s(y)$ of class C^m in which $x_s(0) \in \partial G, \sigma_1$ corresponds to $\partial G \cap N_s$, and G_1 corresponds to $G \cap N_s$ (see §§ 1.1 and A.). Let $u_s = \zeta_s u$. Then $u = u_1 + \ldots + u_S$. If $\Lambda(\zeta_s) \subset G$, then $u_s = 0$ near ∂G and it is easy to see by approximating by mollifiers, etc., that each such $u_s \in H^m_{p0}(G)$ and may be extended to D as in Theorem 2.3.1.

So, suppose $\Lambda(\zeta_s) \subset N_s$, a boundary neighborhood. Let $v_s(y) = u_s[x_s(y)]$. Then $v_s \in H^m_p(G_1)$ and $v_s(y) = 0$ near $\partial G_1 \cap \partial B(0,1)$. If we extend $v_s(y)$ to be 0 for $y \geq 0$ outside G_1, we see that $v_s \in H^m_p(E^+_\nu)$. We shall show how to extend v_s to $\in H^m_{p0}[B(0,1)]$ and to vanish near $\partial B(0,1)$. Then, if we extend u_s to $G \cup N_s$ to be 0 on $D - N_s$ and to be the transform of the extended function v_s on N_s. We shall hereafter drop the subscript s.

If we define $w_n(y) = v(y+n^{-1}e_\nu)$, e being the unit vector in the y^ν direction, we see from Theorem 2.1.4 and Lemma 2.4.1 that $w_n \longrightarrow v$ in $H^m_p(E^+)$ and each w_n vanishes near $\partial G_1 \cap \partial B(0,1)$. For each n, the functions $w_{nq}(y) = v_\rho(y+n^{-1}e_\nu)$ with $\rho = n^{-1}(q+1)^{-1} \in C^\infty_c(E)$ and converge to w_n in $H^m_p(E^+)$ as $q \longrightarrow \infty$ by Theorem 2.1.3. So, if we choose $v_n = w_{n,q_n}$, for q_n sufficiently large, we see that each $v_n \in C^m$ for $y^\nu \geq 0$ and $v_n \longrightarrow v$ in $H^m_p(E^+)$ and v_n vanishes near $\partial G \cap \partial B(0,1)$. If, now, we define

$$v_n(y, y'_\nu) = \sum_{p=1}^{m+1} C_p v_n(-py^\nu, y'_\nu) \; ^{*)} \text{ for } y^\nu \leq 0, \text{ where} \tag{2.4.1}$$

---

*) This device is due to J. L. Lions, see [ ] and

$$\sum_{p=1}^{m+1} (-p)^{q-1} C_p = 1 , \quad q = 1,\ldots, m+1$$

we see that each $v_n \in C^m$ on E, $\Lambda(v_n) \subset B(0,1)$, and that $\| v_n\|^m_{pG^-} \leq C(m,\nu)\cdot \|v_n\|^m_{pG}$ independently of n, so that the $v_n \longrightarrow v$ in H^m_{po} on $B(0,1)$ where $v(y)$ is defined for $y^\nu \leq 0$ by (2.4.1); here $G = G_1$ and $G^- = B(0,1)-G_1-\sigma_1$.

Having extended u to $\in H^m_p(D)$ and to have support in D, we may approximate to u on $\overline{G}$ by the mollified functions u_ρ.

LEMMA 2.4.2: Suppose $u \in H^m_p(D)$, $G \subset D_{2\rho_0}$, $0 < \rho < \rho_0$, φ is a mollifier, and u_ρ is the φ-mollified function of u. Then

$$(2.4.2) \qquad \| u_\rho - u\|^{m-1}_{pG} \leq C\rho \| u\|^m_p , \quad C = C(\nu,m,p,\varphi)$$

Proof: By using the functions u , we may approximate u in $H^m_p(G_{\rho_0})$ by functions $\in C^m(G_{\rho_0})$. Hence we may assume that u $C^m(G_{\rho_0})$. Then, let $v = \nabla^k u$ for $0 \leq k \leq m-1$. Then

$$|v_\rho(x)-v(x)| = |\rho^{-\nu}\int_{B(0,\rho)} \varphi(\rho^{-1}\xi)[\int_0^1 \xi^\alpha \sigma_{,\alpha}(x+t\xi)dt]d\xi|$$

$$(2.4.3) \quad |v_\rho(x)-v(x)|^p \leq K \rho^{p-\nu}\int_{B(0,\rho)}[\int_0^1 |\nabla v(x+t\xi)|^p dt]d\xi$$

where $K = K(\nu,\varphi)$. By integrating (2.4.3) over G, we obtain

$$\|\nabla^k u_\rho - \nabla^k u\|^0_{pG} \leq K\rho \|\nabla^{k+1} u\|^0_{pD} , \quad k = 0, \ldots, m-1$$

from which the result follows.

THEOREM 2.4.2: If G is bounded and of class C^m, then bounded subsets of $H^m_p(G)$ are conditionally compact as subsets of $H^{m-1}_p(G)$. If $u_n \rightharpoonup u$ in $H^m_p(G)$, then $u_n \longrightarrow u$ in $H^{m-1}_p(G)$.

Proof: Let $\{u_n\}$ be a bounded sequence in $H_p^m(G)$. Using Theorem 2.4.1, we may extend each u_n to an open set $D \supset \overline{G}$ so that the extended sequence is bounded in $H_p^m(D)$. Let φ be a mollifier, suppose $\overline{G} \subset D_{\rho_0}$, and $0 < \rho < \rho_0$. Then from Lemma 2.4.2, we conclude that

$$(2.4.4) \qquad \| u_{n\rho} - u_n \|_{pG}^{m-1} \leq M\rho$$

First, let us suppose that $p > 1$. Then $\exists$ a subsequence, still called $\{u_n\}$, such that $u_n \rightharpoonup u$ in $H_p^m(D)$. It follows from the formula (2.1.8) for $u_{n\rho}$ and u_ρ and the weak convergence that $D_\alpha u_{n\rho}$ converges uniformly to $D_\alpha u_\rho$ on $\overline{G}$, for $0 \leq |\alpha| \leq m-1$. Then, taking norms on $H_p^{m-1}(G)$, we obtain

$$(2.4.5) \qquad \| u_n - u \| \leq \| u_n - u_{n\rho} \| + \| u_{n\rho} - u_\rho \| + \| u_\rho - u \|$$

$$\leq 2M\rho + \| u_{n\rho} - u_\rho \|$$

using (2.4.4). The result follows easily. If $p = 1$, all the $D_\alpha u_{n\rho}$ are equicontinuous and uniformly bounded so that a subsequence, still called $\{u_n\}$ exists so that the $D_\alpha u_{n\rho}$ converge uniformly to some functions $\varphi_{\alpha\rho}$ for each of a sequence of $\rho \to 0$. Since (2.4.4) still holds the sequences $\varphi_{\alpha\rho}$ form a Cauchy sequence in $L_p(G)$ and so tend to limit functions φ_α in $L_p(G)$. If we set $u = \varphi_\alpha$ when $|\alpha| = 0$. then we see that $u \in H_p^{m-1}(G)$, $\varphi_\alpha = u_{,\alpha}$, $\varphi_{\alpha\rho} = u_{\rho,\alpha}$ and (2.4.4) holds for u_ρ and u. The result again follows, but now, we do not know that $u \in H_p^m$. However, if $u_n \rightharpoonup u$ in H_1^m, then this u must be the same as the preceding one.

THEOREM 2.4.3: <u>If G is bounded and of class C^m, there are bounded linear mappings B_α for $0 \leq |\alpha| \leq m-1$ of $H_p^m(G)$ into $L_p(\partial G)$ such that $B_\alpha u$ is the restriction of $D_\alpha u$ to ∂G whenever $u \in C^m(\overline{G})$. If $u_n \rightharpoonup u$ in $H_p^m(G)$, then $B_\alpha u_n \to B_\alpha u$ in $L_p(\partial G)$ if $0 \leq |\alpha| \leq m-1$. If $p > 1$, the mappings are compact.</u>

Proof In order to prove the first statement, it is sufficient to show that if $u \in C^m(\overline{G})$, then

(2.4.6) $$\int_{\partial G} |\nabla^k u|^p dS \leq C(\nu, m, p, G)(\| u \|_p^m)^p \qquad 0 \leq k \leq m-1$$

In order to do this, let $\zeta_1, \ldots, \zeta_S$ be a partition of unity as in the proof of Theorem 2.4.1, let $u_s = \zeta_s u$, and let v_s be the transform of u_s under T_s whenever the support of ζ_s is in a boundary neighborhood. Then $\| v_s \|_p^m \leq C_1 \| u \|_p^m$ and

$$\int_{\partial G} |\nabla^k u_s|^p dS \leq C_2 \int_{\sigma_1} \sum_{j=0}^{k} |\nabla^j v_s(0, y'_\nu)|^p dy'_\nu \leq C_3 \int_{G_1} |\nabla^{k+1} v_s(y)|^p dy$$

$$\leq C_4(\| u \|_p^m)^p \quad , \qquad 0 \leq k \leq m-1$$

since $\nabla^k v_s$ vanishes on and near $\partial G_1 \cap \partial B_1$. Thus (2.4.6) follows.

Next, suppose that $u_n \rightharpoonup u$ in $H_p^m(G)$. Using the result above, we may assume that each $u_n \in C^m(G)$. Then, for each s, $u_{ns} \rightharpoonup u_s$ in $H_p^m(G)$ and $v_{ns} \rightharpoonup v_s$ in $H_p^m(G_1)$ (Theorem 2.3.3) so that $v_{ns} \to v_s$ in $H_p^{m-1}(G_1)$. Moreover, we see that

(2.4.7) $$\int_{\sigma_1} |\nabla^k v_{ns}(y^\nu, y'_\nu) - \nabla^k v_{ns}(0, y'_\nu)|^p dy'_\nu \quad , \qquad 0 \leq k \leq m-1$$

$$\leq (y^\nu)^{p-1} \int_{G_1} |\nabla^{k+1} v_{ns}(y)|^p dy \leq \varepsilon(y^\nu) \, , \qquad \lim_{\rho \to 0^+} \varepsilon(\rho) = 0$$

since in the case p = 1, the weak convergence implies the uniform absolute continuity of the integral of $|\nabla^m v_{ns}|$. Since $v_{ns} \to v_s$ in $H_p^{m-1}(G_1)$, each vanishing near $\overline{\Sigma}_1$, it follows that $\nabla^k v_{ns} \to \nabla^k v_s$ in $L_p(\sigma_1)$ for almost all y^ν. Using (2.4.7) it follows that the sequences $\nabla^k v_{ns}(0, y'_\nu)$ are Cauchy sequences in $L_p(\sigma_1)$ so that $\nabla^k u_n$ are Cauchy sequences in $L_p(\partial G)$. Since this holds for any sequence $u_n \ni u_n \rightharpoonup u$ in $H_p^m(G)$, the limit functions being the same, we see that

$\nabla^k u_n \longrightarrow \nabla^k u$ in $L_p(\partial G)$ for $0 \leq k \leq m-1$. If $p > 1$ and $\{u_n\}$ is any bounded sequence in $H_p^m(G)$, a subsequence converges weakly in $H_p^m(G)$ to some u.

2.5 Examples; continuity; some Sobolev lemmas. As was pointed out in the introduction (§1.1), the functions in $H_p^m(G)$ have been studied from many different points of view by many writers. We shall not go deeply into real variable properties of these functions here but will merely give a few examples to indicate the generality of the functions allowed and then shall prove some Sobolev-type lemmas which will indicate when such functions are continuous or have several continuous derivatives.

It is easy to verify that the function f defined by

$$(2.5.1) \qquad f(x) = |x|^{-h} \in H_p^m \text{ on } B(0,1) \text{ iff } (h+m)\cdot p < \nu .$$

From this, it can easily be shown that any function f defined by

$$f(x) = \int_G |x-\xi|^{-h} d\mu(e_\xi) \quad , \qquad (h+m)\cdot p < \nu ,$$

where G is bounded and μ is a finite measure on G, $\in H_p^m$ on any bounded domain D. (2.5.1) shows also that, for a given m and p, the wildness of functions of class H_p^m increases with the dimension ν. We now prove a Sobolev-type lemma guaranteeing continuity.

THEOREM 2.5.1: Suppose $u \in H_p^m(G)$ where G is bounded and of class C^m or is convex and $m > \nu/p$. Then u is continuous on G and there is a constant $C(\nu,m,p,G)$ such that

$$(2.5.2) \qquad |u(x)| \leq C\cdot |G|^{-1/p} \left\{ \sum_{j=0}^{m-1} \frac{\Delta^j}{j!} \|\nabla^j u\|_p^0 + (m-\nu/p)^{-1} \frac{\Delta^m}{(m-1)!} \|\nabla^m u\|_p^0 \right\}.$$

If G is convex, we may take C = 1; u may be a vector function.

<u>Proof</u>: If G is of class C^m, u may be extended to $\in H^m_{po}(D)$ where D is a given open set $\supset G$ with $\|u\|^m_{pD} \leq C_1(\nu, m,p,G,D)\|u\|^m_{pG}$ and then may be extended to be 0 outside D. So, by allowing a factor C, we may replace G by its convex cover. Since u may be approximated in $H^m_p(G_\rho)$ by functions $\in C^m$, for each $\rho>0$, we may assume $G = G_\rho$ and $u \in C^m(G)$ and then let $\rho \to 0$ in (2.5.2).

By expanding in a Taylor series with remainder about each y in G and then integrating with respect to y over G, we obtain (for notation, see §)

$$
\begin{aligned}
|G|\cdot u(x) &= \sum_{j=0}^{m-1} (j!)^{-1}(-1)^j \int_G \nabla^j u(y)\cdot(y-x)^j dy \\
&\quad + \int_0^1 (-1)^m[(m-1)!]^{-1} t^{m-1}\left[\int_G \nabla^m u[x+t(y-x)]\cdot(y-x)^m dy\right]dt
\end{aligned}
\tag{2.5.3}
$$

The inequality follows by applying the Hölder inequality to each term, first setting $z = x + t(y-x)$ to handle the last integral; when this is done the last integral is dominated by

$$[\Delta^m/(m-1)!]\cdot\int_0^1 t^{m-1-\nu}\left[\int_{G(x,t)} |\nabla^m u(z)|dz\right]dt,$$

where $G(x,t)$ consists of all $z=x+t(y-x)$ for $y \in G$ and $|G(x,t)| = t^\nu |G|$.

This shows, for example, that if $u \in H^1_p(G)$ with $p > \nu$, then u is continuous. We now present a "Dirichlet growth" theorem guaranteeing continuity (cf[ ]):

THEOREM 2.5.2: <u>Suppose</u> $u \in H^1_p[B(x_0,R)]$, $1 \leq p \leq \nu$, <u>and suppose</u>

$$\int_{B(x,r)} |u|^p dx \leq L^p(r/\delta)^{\nu-p+p\mu}, \quad 0 \leq r \leq \delta = R-|x-x_0|, \quad 0 < \mu < 1 \tag{2.5.4}$$

<u>for every</u> $x \in B(x_0,R)$. <u>Then</u> $u \in C^0_\mu[B(x_0,r)]$ <u>for each</u> $r < R$ <u>and</u>

$$|u(\xi)-u(x)| \leq CL\delta^{1-\nu/p}(|\xi-x|/\delta)^\mu \quad \text{for } |\xi-x| \leq \delta/2,\ C = C(\nu,p,\mu). \tag{2.5.5}$$

Proof: By approximations, we may assume $u \in C^1(B_R)$. Let x and ξ be given, let $\rho = |\xi - x|/2$, and $\bar{x} = (\xi + x)/2$. For each $\eta \in B(\bar{x},\rho)$, we observe that

$$u(\eta) - u(\xi) = (\eta^a - \xi^a)\int_0^1 u_{,a}[\xi + t(\eta - \xi)]dt$$

$$|u(\eta) - u(\xi)| \le 2\rho \int_0^1 |\nabla u[\xi + t(\eta - \xi)]|dt$$

Averaging over $B(\bar{x},\rho)$, we obtain

$$(2.5.6) \qquad |B(\bar{x},\rho)|^{-1} \int_{B(\bar{x},\rho)} |u(\eta) - u(\xi)|d\eta \le 2\rho |B(\bar{x},\rho)|^{-1}\int_{B(\bar{x},\rho)}\int_0^1 |\nabla u[\xi + t(\eta - \xi)]|dt\,d\eta$$

Interchanging the order of integration, setting $y = \xi + t(\eta - \xi)$ and noting that y ranges over $B(\bar{x}_t, t\rho)$ where $\bar{x}_t = (1-t)\xi + t\bar{x}$ which is at a distance $\delta_t \ge \delta - 2\rho + t\rho \ge \delta/2$ and using (2.5.4) and the Hölder inequality to obtain

$$(2.5.7) \qquad \int_{B(\bar{x}_t, t\rho)} |\nabla u(y)|dy \le C_1 L(\delta/2)^{1-\mu-\nu/p}(t_\rho)^{\nu-1+\mu}, \quad C_1 = C(\nu,p)$$

we see that the right side of (2.5.6)

$$(2.5.8) \qquad \le 2C_2\rho^{1-\nu}L(\delta/2)^{1-\mu-\nu/p}\rho^{\nu-1+\mu}\int_0^1 t^{\mu-1}d\mu, \qquad C_2 = C_2(\nu,p)$$

Using the same result for the average of $|u(\eta) - u(x)|$, we obtain the result.

We now prove another Sobolev Lemma:

THEOREM 2.5.3: Suppose $u \in H_p^1(E_\nu)$ with $1 \le p < \nu$, and has compact support. Then $u \in L_r(E_\nu)$, where $r = \nu p/(\nu - p)$, and

$$(2.5.9) \qquad \|u\|_r^0 \le t \prod_{\alpha=1}^{\nu}(\|u_{,\alpha}\|_p^0)^{1/\nu} \le \nu^{-1/2} t \|\nabla u\|_p^0, \quad 1 \le p < \nu, \quad t = p(\nu-1)/(\nu-p)$$

Proof: It is sufficient to prove this for $u \in C_c^1(E_\nu)$. The case where p>1 can be proved by applying the inequality (2.5.9) for p=1 to the function v defined

by

$$v(x) = |u(x)|^t$$

For, if this is done, we obtain

$$(2.5.10) \qquad \int |u(x)|^r dx = \int |v(x)|^s dx \le \prod_{\alpha=1}^{\nu} (\| v_{,\alpha} \|_1^o)^{s/\nu}$$

$$\le t^s (\| u \|_r^o)^{rs(p-1)/p} \prod_{\alpha=1}^{\nu} (\| u_{,\alpha} \|_p^o)^{s/\nu}, \quad (s = \nu/(\nu-1))$$

since $r = p(t-1)/(p-1)$. The first result in (2.5.9) follows easily since

$$r - [rs(p-1)/p] = s$$

and the second result follows from an elementary inequality.

We shall prove the inequality for $p=1$ and $\nu = 3$; the proof for $\nu = 2$ or $\nu > 3$ is similar. Clearly

$$|u(x,y,z)| \le \int_{-\infty}^{\infty} |u_{,1}(\xi,y,z)| d\xi, \int_{-\infty}^{\infty} |u_{,2}(x,\eta,z)| d\eta, \int_{-\infty}^{\infty} |u_{,3}(x,y,\zeta)| d\zeta$$

Thus

$$|u(x,y,z)|^{3/2} \le \left(\int_{-\infty}^{\infty} |u_{,1}(\xi,y,z)| d\xi \right)^{1/2} \left(\int_{-\infty}^{\infty} |u_{,2}(x,\eta,z)| d\eta \right)^{1/2}$$

$$\left(\int_{-\infty}^{\infty} |u_{,3}(x,y,\zeta)| d\zeta \right)^{1/2}$$

Integrating first with respect to x then y, and using the Schwarz inequality, we obtain

$$\int_{-\infty}^{\infty} |u(x,y,z)|^{3/2} dx \le \left(\int_{-\infty}^{\infty} |u_{,1}(\xi,y,z)| d\xi \right)^{1/2} \left(\iint_{-\infty}^{\infty} |u_{,2}(x,\eta,z)| dx d\eta \right)^{1/2}$$

$$\left(\iint_{-\infty}^{\infty} |u_{,3}(x,y,\zeta)| dx d\zeta \right)^{1/2}$$

$$\iint_{-\infty}^{\infty} |u(x,y,z)|^{3/2} dx dy \le \left(\iint_{-\infty}^{\infty} |u_{,1}(\xi,y,z)| d\xi dy \right)^{1/2} \left(\iint_{-\infty}^{\infty} |u_{,2}(x,\eta,z)| dx d\eta \right)^{1/2}$$

$$\left(\iiint_{-\infty}^{\infty} |u_{,3}(x,y,\zeta)| dx dy d\zeta \right)^{1/2}$$

from which the result follows by integrating with respect to z.

THEOREM 2.5.4: There is a constant $C(\nu)$ such that if $u \in H_2^1[B(x_0,R)]$, there exists a function $U \in H_{20}^1[B(x_0,2R)]$ such that $U(x) = u(x)$ for $x \in B(x_0,R)$ and

$$\|\nabla U\|^0_{2,2R} \le C \cdot {}'\|u\|^1_{2,R} \quad , \text{ where}$$

$$({}'\|u\|^1_{2,R})^2 = \int_{B_R} (|\nabla u|^2 + R^{-2}u^2)dx$$

Proof: From considerations of homogeneity, it follows that it is sufficient to prove this for the unit sphere $B(0,1)$. But then the result follows from Theorem 2.4.1.

THEOREM 2.5.5: (a) If $u \in H_p^1(D)$, $p \ge 1$, then u has a representative $\bar{u}$ such that $u(x^\alpha, x'_\alpha)$ is absolutely continuous in x^α on any segment $[a^\alpha, b^\alpha]$ such that $(x^\alpha, x'_\alpha) \in D$ for $a^\alpha \le x^\alpha \le b^\alpha$, $\alpha = 1,\dots,\nu$. Moreover, its partial derivative $\bar{u}_{,\alpha}$ (which exists almost everywhere and is measurable) is a representative of $u_{,\alpha}$.

(b) Suppose $\bar{u} \in L_p(D)$, $p \ge 1$, suppose $\bar{u}$ has the absolute continuity properties of part (a), and suppose its partial derivatives $\bar{u}_{,\alpha} \in L_p(D)$. There $\bar{u}$ is of class $H_p^1(D)$.

Proof: (a) Let $R = [a,b]$ be any cell in D in which a and b are rational. Suppose $u_n \in C^1(\bar{R})$ and $u_n \to u$ in $H_p^1(R)$. Define

$$\varphi_n(x'_1) = \int_{a^1}^{b^1} [\,|u_n(x^1,x'_1)-u(x^1,x'_1)|^2 + |\nabla u_n(x^1,x'_1)-\nabla u(x^1,x'_1)|^2]^{p/2}dx'$$

wherever this makes sense. Then each $\varphi_n \in L_1([a'_1,b'_1])$ and $\varphi_n \to 0$ in L_1 there. Thus, a subsequence of the φ_n converges to zero for almost all x'_1. For any such x'_1, it is clear that the $u_n(x', x'_1)$ are uniformly A.C. and hence equicontinuous and converge to an AC limit which we denote by $U'(x', x'_1)$ which must

be equivalent to $u(x^1, x_1^{\prime})$ and have partial derivative $U^1_{,1}$ equivalent to $u_{,1}$. By taking successive subsequences for $\alpha = 2, \ldots, \nu$ in turn, we arrive at a function $\bar{u}_R$ having the desired properties on R. But since the totality of rational cells is denumerable, it follows that these functions u_R all coincide almost everywhere with a $\bar{u}$ as desired. The proof of (b) is trivial and is left to the student.

EXERCISE

Prove Theorem 2.5.5(b).

2.6. L_2 properties of potentials and quasi-potentials. In this section, we consider quasi-potentials u defined by

$$(2.6.1) \qquad u(x) = -\int_G K_{0,\alpha}(x-\xi)e^{\alpha}(\xi)d\xi$$

and potentials as defined in § 1.4. If the vector $e \in C_c^{1+\mu}(G)$, then $\Lambda(e) \subset B_R = B(x_0, R)$ for some x_0 and R; we may take $x_0 = 0$. Then we may replace G by B_R in (2.6.1) and we obtain

$$(2.6.2) \qquad u(x) = -\int_G K_0(x-\xi)e^{\alpha}_{,\alpha}(\xi)d\xi \quad ,$$

by integrating by parts. Then $u \in C^{2+\mu}$ everywhere and

$$(2.6.3) \qquad -\Delta u(x) = e^{\alpha}_{,\alpha}(x) \quad , \qquad x \in E_{\nu}.$$

Thus, if D is any open set and $v \in H^1_{2o}(D)$, we see by approximations to v by smooth functions that

$$(2.6.4) \qquad \int_D \bar{v}_{,\alpha}(u_{,\alpha} + e^{\alpha})dx = 0 \quad , \qquad v \in H^1_{2o}(D).$$

Moreover, from (2.6.3), we see that

$$(2.6.5) \qquad \int_{B_A} |\nabla u(x)|^2 dx = +\int_{\partial B_A} \bar{u}\frac{\partial u}{\partial r}dS - \int_G e^{\alpha}\bar{u}_{,\alpha}dx \quad , \qquad G \subset B_A$$

From the formula (2.6.1), we see that $\bar{u} = O(A^{1-\nu})$, $\partial u/\partial r = O(A^{-\nu})$, so that we may let $A \longrightarrow \infty$ in (2.6.5) and conclude that $\nabla u \in L_2(E_\nu)$ and that

$$(2.6.6) \qquad \|\nabla u\|_2^0 \leq \|e\|_2^0$$

LEMMA 2.6.1: If S is any bounded set and $0 < h < \nu$,

$$\Gamma_\nu^{-1} \int_S |x|^{-h} dx \leq (\nu - h)^{-1} r^{\nu-h} , \qquad \text{where } \gamma_\nu r^\nu = |S|$$

<u>Proof:</u> For the integral on the left is not decreased if S is replaced by $B(0,r)$ which has the same measure.

THEOREM 2.6.1: <u>Suppose</u> G <u>is bounded</u>, $e \in L_2(G)$, <u>and</u> u <u>is defined by</u> (2.6.1) <u>whenever this makes sense</u>. <u>Then</u> $u \in H_2^1(D)$ <u>and satisfies</u> (2.6.4) <u>for any bounded open set</u> D. <u>Finally</u> $\nabla u \in L_2(E_\nu)$ <u>and</u> (2.6.6) <u>holds</u>. <u>If</u> $D = G = B_R$, <u>then</u>

$$(2.6.7) \qquad \|u\|_2^0 \leq R \|e\|_2^0$$

<u>Proof:</u> If $e \in L_2(G)$, and D is bounded, we see that $|\xi - x|^{1-\nu} \cdot |e(\xi)|^2$ is 2ν dimensionally summable over $D \times G$ and so it is summable in ξ over G for almost all x. Since $K_{0,\alpha}(x-\xi)e^\alpha(\xi)$ is 2ν-dimensionally measurable, it is measurable in ξ for almost all x. Thus, for almost all x, $u(x)$ is defined and

$$(2.6.8) \qquad \begin{aligned} |u(x)|^2 &\leq \left\{\Gamma_\nu^{-1}\int_G |\xi - x|^{1-\nu} d\xi\right\} \cdot \left\{\Gamma_\nu^{-1}\int_G |\xi - x|^{1-\nu} |e(\xi)|^2 d\xi\right\} \\ &\leq \gamma_\nu^{-1/\nu} \cdot |G|^{1/\nu} \cdot \Gamma_\nu^{-1}\int_G |\xi - x|^{1-\nu} \cdot |e(\xi)|^2 d\xi \end{aligned}$$

by the Schwarz Inequality and Lemma 2.6.1. Integrating (2.6.8) we see that $u \in L_2(D)$ and

$$(2.6.9) \qquad \|u\|_{2D}^0 \leq \gamma_\nu^{-1/\nu} |G|^{1/2\nu} |D|^{1/2\nu} \|e\|_2^0$$

(2.6.7) follows from (2.6.9) if $D = G = B_R$.

Now, we may approximate to any $e \in L_2(G)$ by a sequence e_n , each $\in C_c^{1+\mu}(G)$. Let u_n be the corresponding quasi-potential and let D be any bounded domain. Then, from (2.6.8), we see that $u_n \longrightarrow u$ in $L_2(D)$ and, from (2.6.6) for each n, the ∇u_n form a Cauchy sequence in $L_2(D)$. It follows that $u \in H_2^1(D)$ and that $\nabla u_n \longrightarrow \nabla u$ in L_2 . Thus, we may pass to the limit in (2.6.4).

THEOREM 2.6.2: _If_ G _and_ D _are bounded_, $f \in L_2(G)$, _and_ u _is its potential as defined by_

(2.6.10) $\quad u(x) = \int_G K_0(x-\xi)f(\xi)d\xi$,

then $u \in H_2^2(D)$ _and_

(2.6.11) $\quad +\Delta u(x) = f(x)$, $\quad u_{,\alpha}(x) = \int_G K_{0,\alpha}(x-\xi)f(\xi)d\xi$

almost everywhere. _If_ $G = D = B_R$, _then_

(2.6.12)
$$\|\nabla u\|_2^0 \le R\|f\|_2^0 , \quad \|\nabla u\|_2^0 \le \|f\|_2^0 ,$$
$$\|u\|_2^0 \le 2^{-1}(\nu-2)^{-1}R^2\|f\|_2^0 \quad \text{if } \nu > 2$$
$$\le 2^{-1}R^2(2^{-1}\log R)\|f\|_2^0 \quad \text{if } \nu = 2 \text{ and } R \le 1/2 .$$

Proof: By proceeding as in (2.6.8), it is easy to verify the bounds for u. For smooth functions f, (2.6.11) holds. For such functions, we see by treating the right side of the second formula in (2.6.11), that

$$|\nabla u(x)| \le \int_G |\nabla K_0(x-\xi)||f(\xi)|d\xi = \Gamma_\nu^{-1}\int_G |\xi - x|^{1-\nu}|f(\xi)|d\xi$$

and we can obtain the bound (2.6.9) with u replaced by ∇u. Since the right sides converge in L_2 , the second formula in (2.6.11) holds.

If $f \in C_c^{1+\mu}(G)$, then

$$+\Delta u(x) = f(x), \qquad +\Delta u_{,\alpha}(x) = f_{,\alpha}(x)$$

Multiplying through by $\bar{u}_{,\alpha}$ and summing and integrating, we obtain

$$(2.6.13) \qquad -\int_{B_A} \bar{u}_{,\alpha} \Delta u_{,\alpha} dx = -\int_{\partial B_A} \bar{u}_{,\alpha} \frac{\partial u_{,\alpha}}{\partial r} dS + \int_{B_A} |\nabla^2 u|^2 dx$$

$$= -\int_{B_A} \bar{u}_{,\alpha} f_{,\alpha} dx = -\int_{B_A} f \Delta \bar{u} dx = \int_G |f|^2 dx$$

By letting $A \longrightarrow \infty$, we see that $\|\nabla^2 u\|_{2E}^0 \leq \|f\|_{2G}^0$. The remaining results follow easily by approximations.

COROLLARY: <u>If</u> $e \in H_{2o}^1(G)$, $G \subset B(x_o, R)$, <u>and</u> u <u>is its quasi-potential, then</u> u <u>is the potential of</u> $e^\alpha_{,\alpha}$ <u>and</u>

$$\|u\|_2^0 \leq R\|e\|_2^0 \leq 2^{-1/2} R^2 \|\nabla e\|_2^0, \quad \|\nabla u\|_2^0 \leq 2^{-1/2} R \|\nabla e\|_2^0,$$

$$\|\nabla^2 u\|_2^0 \leq \|e^\alpha_{,\alpha}\|_2^0 \leq \|\nabla e\|_2^0,$$

The results follow from the theorems above and Poincaré's inequality; that $\|e^\alpha_{,\alpha}\|_2^0 \leq \|\nabla e\|_2^0$ follows by taking Fourier transforms.

2.7. <u>Miscellaneous additional inequalities.</u> In this section we prove additional inequalities which are especially useful in discussing equations of higher order.

THEOREM 2.7.1: <u>Suppose</u> G <u>is bounded and of class</u> C^m, $m > 1$ <u>and</u> $p \geq 1$. <u>Then, for each</u> j, $1 \leq j \leq m-1$, <u>and each</u> $\varepsilon > 0$, <u>there is a constant</u> $C(\nu, m, p, G, j, \varepsilon)$ <u>such that</u>

$$(2.7.1) \qquad \|\nabla^j u\|_p^0 \leq \varepsilon \|\nabla^m u\|_p^0 + C\|u\|_p^0, \qquad u \in H_p^m(G).$$

Proof: For otherwise there is a sequence $\{u_n\}$ with $u_n \in H_p^m(G)$ and $\|u_n\|_p^m = 1$ such that

$$(2.7.2) \qquad \|\nabla^j u_n\|_p^o > \varepsilon\|\nabla^m u_n\|_p^o + n\|u_n\|_p^o .$$

From Theorem 2.4.2, it follows that we may assume that $u_n \longrightarrow u$ in $H_p^{m-1}(G)$. Since $\|\nabla^j u_n\|_p^o \leq \|u_n\|_p^{m-1} \leq \|u_n\|_p^m = 1$, it follows that $u_n \longrightarrow u$ in $L_p(G)$, so $u = 0$. But then $\|\nabla^j u_n\|_p^o \longrightarrow 0$ so that $\|\nabla^m u_n\|_p^o$ and hence $\|u_n\|_p^m \longrightarrow 0$.

THEOREM 2.7.2: If $u \in H_p^m(G)$ and G is bounded, there is a unique polynomial P of degree $\leq m-1$ (or $= 0$) such that the average over G of each $D_\alpha(u-P)$ is 0 if $0 \leq |\alpha| \leq m-1$.

This is easily proved by induction on m.

THEOREM 2.7.3: Suppose G is a domain of class C^m and $G \subset B(x_o, R)$. Then there is a constant $C(\nu, m, p, G)$ such that

$$(2.7.3) \qquad \|\nabla^j u\|_p^o \leq CR^{m-j}\|\nabla^m u\|_p^o \quad , \qquad 0 \leq j \leq m-1$$

for every $u \in H_p^m(G)$ such that the average over G of each $D_\alpha u$ is 0 for $0 \leq |\alpha| \leq m-1$.

Proof: We first assume that $G \subset B(0,1)$. Suppose the theorem is not true. Then there is a j, $0 \leq j \leq m-1$, and a sequence $\{u_n\}$ of functions of the type described such that

$$(2.7.4) \qquad \|\nabla^j u_n\|_p^o > n\|\nabla^m u_n\|_p^o \quad , \qquad \|u_n\|_p^m = 1$$

A subsequence, still called $\{u_n\}$, converges in $H_p^{m-1}(G)$ to some u. From (2.7.4), it follows that $\nabla^m u_n \longrightarrow 0$ in $L_p(G)$ so that $u_n \longrightarrow u$ in $H_p^m(G)$ and $\nabla^m u = 0$. By induction on m and the use of mollifiers, it can be shown that u is a polynomial of degree $\leq m-1$. Clearly the averages over G of each $D_\alpha u$ is 0 if $0 \leq |\alpha| \leq m-1$. From Theorem 2.7.2, it follows that $u=0$. Since $u_n \longrightarrow u$ in $H_p^m(G)$, this contradicts (2.7.4).

CHAPTER 3

THE GENERAL SECOND ORDER ELLIPTIC EQUATION WITH REAL COEFFICIENTS.

3.1. The H_2^1 existence theory. In this chapter, we discuss the existence and differentiability of the solutions of the equation

(3.1.1) $\quad Lu + \lambda u = f$ where $Lu = -(a^{\alpha\beta}u_{,\alpha\beta} + b^\alpha u_{,\alpha} + cu)$, $u = 0$ on ∂G

where the $a^{\alpha\beta}$, b^α, and c are real but u and f may be complex. The coefficients $a^{\alpha\beta}$ are always assumed continuous on the closure of the domain being considered and we assume that there is an h such that

(3.1.2) $$(1 - h)|\lambda|^2 \leq a^{\alpha\beta}(x)\lambda_x\lambda_\beta \leq (1 + h)|\lambda|^2 \text{ and } 0 < h < 1 ,$$

for all x on $\bar{G}$ and all λ. We assume that the b^α and c are bounded and measurable at least.

Our method of proof for the case that the $a^{\alpha\beta}$ are Lipschitz is as follows: Suppose u is a solution of (3.1.1) which $\varepsilon H_2^2(D)$ for each $D \subset\subset G$. Then if $v \varepsilon C_c^\infty(G)$, we obtain by integrating by parts, the result that

(3.1.3) $$\int_G \{ \bar{v}_{,\alpha} a^{\alpha\beta} u_{,\beta} + \bar{v}(a^{\alpha\beta}_{,\alpha} u_{,\beta} - b^\alpha u_{,\alpha} - cu + du)\} \lambda x = \int_G f \bar{v} dx .$$

We shall then show that there is a unique solution $u \varepsilon H_{20}^1(G)$ of (3.1.3) for each f in L_2 provided λ is not in an isolated set of eigenvalues. We then show that $u \varepsilon H_2^2(G)$ if G is of class C_1^1 and obtain further differentiability properties of the solutions which hold when G, f, and the coefficients satisfy additional differentiability conditions.

The equation (3.1.3) is a special case of the equation

(3.1.4) $$B(u, v) + \lambda C(u, v) = L(v), \ v \ \varepsilon \ H_{20}^1(G), \text{ where}$$

(3.1.5) $$B(u, v) = \int_G [\bar{v}_{,\alpha}(a^{\alpha\beta}u_{,\beta} + b^\alpha u) + \bar{v}(c^\alpha u_{,\alpha} + du)]dx ,$$
$$C(u, v) = \int_G u\bar{v}dx , \ L(v) = -\int_G (e^\alpha \bar{v}_{,\alpha} + f\bar{v}) dx ,$$

which we consider for the remainder of this section. In this section we shall begin by assuming merely that the $a^{\alpha\beta}$ are bounded and measurable and satisfy (3.1.2). We shall use the norm and inner product of (2.1.15) for $m = 1$ throughout this section. We denote the bounds for b, c, and d in (3.1.5) by

(3.1.6) $$|b(x)| \leq B_1, \quad |c(x) \leq C_1, \quad |d(x)| \leq D ,$$

THEOREM 3.1.1: <u>If</u> $u, v \in H^1_{20}(G)$ <u>and</u> G <u>is bounded,</u>

(3.1.7) $$\operatorname{Re}B(u, u) \geq 2^{-1}(1 - h)(\|u\|)^2 - \lambda_0 C(u, u) \quad ,$$

(3.1.8) $$|B(u, v)| \leq M_1 \|u\| \cdot \|v\| , \quad (\|\varphi\| = \|\varphi\|^1_{20})$$

<u>where</u>

(3.1.9) $$\lambda_0 = D_1 + 2^{-1}(1 - h)^{-1}(B_1 + C_1)^2 , \quad M_1 = (1 + h) + 2^{-1}(B_1+C_1)R+2^{-2}D_1R^2$$

<u>if</u> $G \subset F(x_0, R)$.

<u>Proof.</u> For

$$|\textstyle\int_G a^{\alpha\beta} u_{,\beta} \bar{v}_{,\alpha} dx| \leq (1 + h)\|u\| \cdot \|v\|$$

$$(1 - h)\|u\|^2 \leq \textstyle\int_G a^{\alpha\beta} u_{,\beta} \bar{u}_{,\alpha} dx \leq (1 + h)\|u\|^2$$

$$|du\bar{v}| \leq D_1 |u| \cdot |v| , \quad |b^\alpha u \bar{v}_{,\alpha}| \leq B_1 |u| \cdot |\nabla v|, \quad |c^\alpha u_{,\alpha} \bar{v}| \leq C_1 |\nabla u| \cdot |v|$$

(3.1.10)

$$|b^\alpha u \bar{u}_{,\alpha} + c^\alpha u_{,\alpha} \bar{u}| \leq 2^{-1}[(1 - h)|\nabla u|^2 + (1 - h)^{-1}(B_1 + C_1)^2 |u|^2]$$

The results follow by integrating (3.1.10) and using the Schwarz and Poincaré (Theorem 2.1.5) inequalities.

THEOREM 3.1.2 (Lemma of Lax and Milgram): <u>Suppose, in a Hilbert space</u> $\mathcal{H}$, $B_0(u, v)$ <u>is linear in</u> u <u>for each</u> v <u>and conjugate linear in</u> v <u>for each</u> u <u>and suppose</u>

(i) $$|B_0(u, v)| \leq M_1 \|u\| \cdot \|v\|$$

(3.1.11)

(ii) $$|B_0(u, u)| \geq m_1 \cdot \| u \|^2, m_1 > 0 .$$

Suppose the transformation T_0 is defined by the condition

(3.1.12) $$B_0(u, v) = (T_0 u, v).$$

Then T_0 and T_0^{-1} are operators with bounds M_1 and m_1^{-1}, respectively.

Proof: It is clear that T_0 is a linear operator with bound M_1. From (3.1.11) (ii) and (3.1.12), we see that

$$m_1 \| u \|^2 \leq |B_0(u, u)| = |(T_0 u, u)| \leq \| u \| \cdot \| T_0 u \|$$

so that

$$\| T_0 u \| \geq m_1 \| u \|$$

It follows easily that the range of T_0 is closed. If the range were not the whole space, there would be a v such that $B_0(u, v) = (T_0 u, v) = 0$ for every u. But, by setting $u = v$, it follows from (ii) that $v = 0$. Thus T_0^{-1} is a bounded operator with norm $\leq m_1^{-1}$.

THEOREM 3.1.3: Suppose the transformation U is defined on $H_{20}^1(G)$ by the condition that

(3.1.13) $$C(u, v) = (Uu, v)_{20}^1, \quad v \varepsilon H_{20}^1(G) .$$

Then U is a completely continuous operator.

Proof. That U is an operator follows from Poincaré's inequality, since

$$\| Uu \| = \sup (Uu, v) = \sup \int_G u \bar{v} \, dx \leq 2^{-1} R^2 \| u \| \text{ if } \| v \| = 1 .$$

Next, suppose $u_n \longrightarrow u$ in $H_{20}^1(G)$. Then $u_n \longrightarrow u$ in L_2 and

$$\| U(u_n - u) \| = \sup_v \int_G (u_n - u) \bar{v} \, dx \leq 2^{-1/2} R \| v \| \cdot \| u_n - u \|_0 \longrightarrow 0$$

so that U is compact.

THEOREM 3.1.4: If λ is not in a set $\mathcal{E}$, which has no limit points in the plane, the equation (3.1.4) has a unique solution u in $H^1_{20}(G)$ for each given e and f in $L_2(G)$. If $\lambda \in \mathcal{E}$, there are solutions of (3.1.4) in which $u \neq 0$ and $e = f = 0$, but the manifold of these is finite dimensional. If λ_0 is defined as in Theorem (3.1.1), then no real number $\lambda_1 > \lambda_0$ is in $\mathcal{E}$.

Proof. Let us define λ_0 as in Theorem 3.1.1 and B_0 by

$$B_0(u, v) = B(u, v) + \lambda_0 C(u, v) \tag{3.1.14}$$

and define T_0 by (3.1.12). Then, equation (3.1.4) is equivalent to

$$T_0 u + (\lambda - \lambda_0) Uu = w \text{ , where } (w,v) = L(v) \text{ .} \tag{3.1.15}$$

Moreover, from Theorem 3.1.1, it follows that B_0 satisfies the conditions of the Lemma of Lax and Milgram with $m_1 = 2^{-1}(1 - h)$. Accordingly T_0 has a bounded inverse so (3.1.15) is equivalent to

$$u + (\lambda - \lambda_0) T_0^{-1} Uu = T_0^{-1} w \text{ .} \tag{3.1.16}$$

Since $T_0^{-1}U$ is compact, the theorem follows from the Riesz theory of linear operators.

3-2. H^2_2 differentiability of the solutions. In this section, we first prove boundedness and approximation theorems and then show that the solution u obtained in § 3.1 $\in H^2_2$.

THEOREM 3.2.1: Suppose the coefficients, e, and f satisfy the conditions of § 3.1 on G; the $a^{\alpha\beta}$ need not be continuous. Suppose $u \in L_2(G)$, $u \in H^1_2(D)$ for each $D \subset\subset G$, and suppose u is a solution of (3.1.4) with $\lambda = 0$. Then there is a constant C, depending only on h, B_1, C_1, D_1, and δ_0 such that if $D \subset\subset G$ and $D \subset G_\delta$, then

$$\|\nabla u\|^0_{2,D} \leq C[\delta^{-1}\|u\|^0_{2,G} + \|e\|^0_{2,G} + \delta\|f\|^0_{2,G}], \quad 0 < \delta \leq \delta_0 \text{ ,} \tag{3.2.1}$$

Proof. In the equation (3.1.4) with $\lambda = 0$, we transpose $L(v)$ to the left side, set $v = \zeta^2 u$, and take the real part, where $\zeta(x) = 1$ if $x \varepsilon D$, $\zeta(x) = 1 - 2\delta^{-1} d(x, D)$ if $0 \leq d(x, D) \leq \delta/2$, $\zeta(x) = 0$ elsewhere. Then, since $a^{\alpha\beta} = a^{\alpha\beta}$ and the $a^{\alpha\beta}$ are real, we obtain

$$(3.2.2) \quad \begin{aligned} 0 = \operatorname{Re} B(U, U) + \operatorname{Re} \int_G \{ \zeta e^\alpha \overline{U}_{,\alpha} - u\overline{u}\,[a^{\alpha\beta} \zeta_{,\alpha}\zeta_{,\beta} + (c^\alpha - b^\alpha)\zeta\zeta_{,\alpha}] \\ + \zeta\overline{u}\,(e^\alpha \zeta_{,\alpha} + \zeta f)\}\, dx \end{aligned}$$

where we have set

$U = \zeta u$, so that $v_{,\alpha} = \zeta(U_{,\alpha} + \zeta_{,\alpha} u)$, $\zeta u_{,\alpha} = U_{,\alpha} - \zeta_{,\alpha} u$, etc.

Since $0 \leq \zeta(x) \leq 1$, $\zeta \varepsilon C_1^0(\overline{G})$, and $|\nabla \zeta(x)| \leq 2\delta^{-1}$, we see from Theorem 3.1.1 that (norms in $L_2(G)$)

$$0 \geq m_1 \|\nabla U\|^2 - \|e\| \cdot \|\nabla U\| - \|u\|^2[\lambda_0 + 4(1+h)\delta^{-2} + 2(B_1 + C_1)\delta^{-1}] - \|u\|(2\delta^{-1}\|e\| + \|f\|).$$

The result follows since

$$\|\nabla u\|_D = \|\nabla U\|_D \leq \|\nabla U\|_G \ .$$

THEOREM 3.2.2: Suppose $G = G_R$, suppose the coefficients and e and f satisfy the conditions of Theorem 3.2.1 on G, suppose $u \varepsilon L_2(G)$, $u \varepsilon H_2^1(G_r)$ for each $r < R$, suppose $u = 0$ along σ_R, and suppose u is a solution of (3.1.4) with $\lambda = 0$. There is a constant C, depending only on h, B_1, C_1, and D_1, such that

$$(3.2.3) \qquad \|\nabla u\|_{2,r}^0 \leq C[(R-r)^{-1}\|u\|_{2,R}^0 + \|e\|_R^0 + (R-r)\|f\|_R^0]$$

Proof: The proof is identical with the preceding proof except that we define $\zeta(x) = 1$ if $|x| \leq r$, $\zeta(x) = 1 - 2(R-r)^{-1}(|x| - r)$ if $r \leq |x| \leq (r+R)/2$, $\zeta(x) = 0$ if $|x| \geq (R+r)/2$. Then u, v, and U all vanish on ∂G_R.

THEOREM 3.2.3: Suppose G is bounded and suppose that the coefficients

$a_n^{\alpha\beta}$, b_n^α, c_n^α, and d_n all satisfy the conditions of Theorem 3.2.1 uniformly on G and converge almost everywhere to $a^{\alpha\beta}$, b^α, c^α, and d, respectively, and suppose that $e_n^\alpha \to e^\alpha$ and $f_n \to f$ in $L_2(G)$. Suppose that $u_n \rightharpoonup$ in $H_2^1(G)$ and that u_n is a solution of $(3.1.4)_n$ for each n. Then u is a solution of (3.1.4) with the limiting functions.

Proof: For each fixed $v \in H_{20}^1(G)$, we see that

$$a_n^{\alpha\beta}\bar{v}_{,\alpha} \to a^{\alpha\beta}\bar{v}_{,\alpha}, \quad b_n^\alpha\bar{v}_{,\alpha} \to b^\alpha\bar{v}_{,\alpha}, \text{ etc.}$$

in $L_2(G)$, so that

$$B_n(u_n, v) \to B(u, v), \; C(u_n, v) \to C(u,v), \; L_n(v) \to L(v) .$$

LEMMA 3.2.1: If $\varphi \in H_p^1(G)$, $D \subset\subset G$, $\bar{D} \subset G_{2H}$, e_γ is the unit vector in the x^γ direction, and φ_h is defined on D for $0 < h < H$ by

$$\varphi_h(x) = h^{-1}[\varphi(x + he_\gamma) - \varphi(x)] ,$$

then $\varphi_h \to \varphi_{,\gamma}$ in $L_p(D)$.

Proof. We approximate strongly in $H_p^1(G_H)$ to φ by functions φ_n of class C' on $\bar{G}_H$. The remainder of the proof is like that of Lemma 2.4.1.

THEOREM 3.2.4: Suppose G is bounded, suppose the coefficients and e and f satisfy the previous conditions on G, and suppose that $u \in H_2^1(G)$ and satisfies (3.1.4) on G. Suppose also that $a^{\alpha\beta}$ and $b^\alpha \in C_1^0(D)$ and $e \in H_2^1(D)$ for each $D \subset\subset G$. Then $u \in H_2^2(D)$ for each $D \subset\subset G$ where it satisfies (almost everywhere) the differential equation

$$(3.2.4) \qquad -\frac{\partial}{\partial x^\alpha}(a^{\alpha\beta}u_{,\beta} + b^\alpha u + e^\alpha) + c^\alpha u_{,\alpha} + du + f = 0 .$$

If G is of class C_1^1, $a^{\alpha\beta}$ and $b^\alpha \in C_1^0(\bar{G})$, and $e^\alpha \in H_2^1(G)$, and $u \in H_{20}^1(G)$, then $u \in H_2^2(G)$.

Proof. Let $D \subset\subset G$ and choose D' and D'' so that $D \subset\subset D' \subset\subset D'' \subset\subset G$.

Let U be the D''-potential of $c^\alpha u_{,\alpha} + du + f$ and let

$$E^\alpha = b^\alpha u + e^\alpha - U_{,\alpha} \quad .$$

Then (3.1.4) is equivalent to

$$(3.2.5) \qquad \int_{D''} v_{,\alpha} (a^{\alpha\beta} u_{,\beta} + E^\alpha) dx = 0 \qquad \text{for } v \in H^1_{20}(D'')$$

and $E^\alpha \in H^1_2(D'')$. We suppose $D' \subset\subset D''_H$.

Now, suppose $v \in C^\infty_c(D')$, e_γ is the unit vector in the x^γ direction. For $0 < |h| < H$, we define

$$v_h(x) = h^{-1}[v(x - he_\gamma) - v(x)] \ , \ u_h(x) = h^{-1}[u(x + he_\gamma) - u(x)] \ , \ x \in D' \ .$$

Then, by writing out what v_h is and changing variables as indicated, we obtain

$$(3.2.6) \quad \begin{aligned} &\int_{D''} v_{h,\alpha}(x)(a^{\alpha\beta} u_{,\beta} + E^\alpha) dx = \int_{D'} v_{,\alpha}(x)[a^{\alpha\beta}(x) u_{h,\beta}(x) + E^\alpha_h(x) dx = 0 \\ &E^\alpha_h(x) = h^{-1}\{[a^{\alpha\beta}(x + he_\gamma) - a^\alpha(x)] u_{,\beta}(x + he_\gamma) + E^\alpha(x + he_\gamma) - E^\alpha(x)]\} \end{aligned}$$

From the bounded convergence of $h^{-1}[a^{\alpha\beta}(x + he_\gamma) - a^\alpha(x)]$ to $a^{\alpha\beta}_{,\gamma}(x)$ and from Lemmas 2.4.1 and 3.2.1, it follows that

$$E^\alpha_h \longrightarrow a^{\alpha\beta}_{,\gamma} u_{,\beta} + E^\alpha_{,\gamma} \text{ in } L_2(D') .$$

From Lemma 3.2.1, it follows that $u_h \longrightarrow u_{,\gamma}$ in $L_2(D')$. Then it follows from Theorem 3.2.1 that $\|\nabla u_h\|^0_{2,D}$ is uniformly bounded for $0 < |h| < H$ so that u_h —— u^* in $H^1_2(D)$ for some sequence of $h \longrightarrow 0$. But u^* must be $u_{,\gamma}$ so that $u_{,\gamma} \in H^1_2(D)$. Since this is true for any γ and D , $u \in H^2_2(D)$. Then the differential equation (3.2.4) follows by integrating (3.1.4) by parts on each D .

In order to prove the last statement, we pick any point x_0 on ∂G and map a boundary neighborhood N of x_0 in the proper way onto G_R by a regular map of class C^1_1 . Then (3.1.4) goes over into an equation of the same form on G_R . We may repeat the argument of the preceding paragraphs for each $\gamma \le \nu - 1$,

since each u_h vanishes along $\sigma_{r'}$. Using Theorem 3.2.2 this time, we conclude that each $u_{,\gamma}$ with $\gamma \leq \nu - 1 \varepsilon H_2^1(G_r)$ for each $r < R$. This implies that all $u_{,\gamma\delta}$ with $\gamma \leq \nu - 1$ and $1 \leq \delta \leq \nu \varepsilon L_2(G_r)$. But now if $x^\nu > \varepsilon > 0$, $u_{,\nu}$ is also in H_2^1 and u satisfies (3.2.4). Since $a^{\nu\nu} > 0$, we can solve that equation for $u_{,\nu\nu}$ and thus conclude that it $\varepsilon L_2(G_r)$ also so that $u_{,\nu}$ also $\varepsilon H_2^1(G_r)$ so that $u \varepsilon H_2^2(G_r)$. Thus, since each point x_0 of ∂G is in a neighborhood on which $u \varepsilon H_2^2$, we conclude that $u \varepsilon H_2^2(G)$.

COROLLARY 1: <u>If, for some boundary neighborhood N of x_0 on $G \cup \partial G$, the part $N \cap \partial G$ is of class C_1^1 and if $a^{\alpha\beta}$ and $b^\alpha \varepsilon C_1^0(N)$ and $e^\alpha \varepsilon H_2^1(N)$, and if u is the solution of (3.1.4) in $H_{20}^1(G)$, then $u \varepsilon H_2^2(N')$ for some boundary neighborhood N' of x_0 with $N' \subset N$.</u>

THEOREM 3.2.5: <u>Suppose G is of class C_1^1 , $a^{\alpha\beta} \varepsilon C_1^0(\bar{G})$, b^α and c are bounded and measurable on G , and the $a^{\alpha\beta}$ satisfy (3.1.2) on $\bar{G}$. Then, if λ is not in a set $\mathcal{C}$ without finite limit points, there is a unique solution of (3.1.1) which $\varepsilon H_2^2(G) \cap H_{20}^1(G)$. If $\lambda \varepsilon \mathcal{C}$, there is a non-empty but finite dimensional manifold of solutions in $H_2^2(G) \cap H_{20}^1(G)$ with $f = 0$.</u>

<u>3.3</u> <u>The equation (3.1.1).</u> Theorem 3.2.5 sums up the results so far obtained concerning this equation. An essential hypothesis is that the $a^{\alpha\beta} \varepsilon C_1^0(G)$. In this section, we replace this hypothesis with the weaker one that the $a^{\alpha\beta} \varepsilon C^0(\bar{G})$. We continue to assume that the b^α and $c \varepsilon M(G)$. We shall also assume that G is of class C_1^1 .

LEMMA 3.3.1: <u>Suppose $G = B(0, R_0)$ or G_{R_0} and $a^{\alpha\beta}(0) = \delta^{\alpha\beta}$. Then there exists an R_1 with $0 < R_1 \leq R_0$, which depends only on ν, h, the bounds for the coefficients and the moduli of continuity of the $a^{\alpha\beta}$, such that</u>

$$'\|u\|_{2,R}^2 \leq 2\|Lu\|_{2,R}^0 \quad \text{if} \quad u \in H_2^2 \quad \text{and} \quad 0 < R \leq R_1$$

<u>if</u> $\Lambda(u) \subset B_R$ <u>or</u> u <u>vanishes along</u> σ_R <u>and near</u> Σ_R. <u>Here</u>

(3.3.1) $$'(\|u\|_{2,R}^2)^2 = \int_{B_R}(|\nabla^2 u|^2 + R^{-2}|\nabla u|^2 + R^{-4}u^2)dx$$

<u>Proof</u>: We first prove this for the case B_R. It is sufficient to prove it for u of class $C_c^2(B_R)$. Then, as in Section 1.3,

$$u(x) = -\int_{B_R} K_0(x-\xi)\Delta u(\xi)\,d\xi .$$

Thus, from Theorem 2.6.2, it follows that $(\|\varphi\| = \|\varphi\|_{2,R}^0)$

$$'\|u\|_{2,R}^2 \leq \|-\Delta u\| \leq \|Lu\| + \|(a-a_0)\cdot\nabla^2 u + b\cdot\nabla u + cu\|$$

$$\leq \|L\,u\| + [\varepsilon(R) + B_1 R + C_1 R^2]'\,\|u\|_{2,R}^2 , \quad \lim_{R\to 0} \varepsilon(R) = 0$$

The result for B_R follows. Now if $u \in H_2^2(G_R)$, vanishes along σ_R and near Σ_R, then it is easy to see by using mollifiers of the form $\varphi(x) = \varphi(|x|)$ that $u \in H_2^2(B_R)$ if it is extended to the part $x^\nu \leq 0$ of B_R by reflection

(3.3.2) $$u(x^\nu, x'_\nu) = -u(-x^\nu, x'_\nu)$$

Since Δu is extended by the same formula, the result follows for G_R.

THEOREM 3.3.1: <u>Under our general hypotheses, there is a constant</u> C <u>which depends only on</u> ν, h, G, <u>the bounds for the coefficients, and the moduli of continuity of the</u> $a^{\alpha\beta}$, <u>such that</u>

(3.3.3) $$\|u\|_2^2 \leq C\,[\,\|Lu\|_2^0 + \|u\|_1^0\,]$$

for all $u \in H_2^2(G)$ which vanish on ∂G.

<u>Proof</u>: First, let $x_0 \in G$. By an affine transformation with upper and lower bounds depending only on h, we may carry x_0 into the origin and a neighborhood of x_0 into B_{R_0} in such a way that the new $'a^{\alpha\beta}(0) = \delta^{\alpha\beta}$. If $x_0 \in \partial G$, we begin with an affine transformation as above, follow that by a rotation of axes so that the tangent plane at x_0 is the plane $x^\nu = 0$ with

x axis pointing into G, and then may follow this by a map of the form $'x^\alpha = x^\alpha$ for $\alpha < \nu$, $'x^\nu = x^\nu - f(x'_\nu)$ which flattens out a part of ∂G near x_0; the result is a regular map of class C_1' of a boundary neighborhood of x_0 onto G_{R_0} so that $'a^{\alpha\beta}(0) = \delta^{\alpha\beta}$. It follows from Lemma 3.3.1 that there is a constant C_1 depending only on h and that each point x_0 of $G \cup \partial G$ is in a neighborhood or boundary neighborhood $N(x_0)$, which depends only on the quantities stated, such that if $\Lambda(u) \subset N(x_0)$ (meaning u vanishes near $\partial N \cap G$ and along $\partial G \cap N$ if N is a boundary neighborhood) then

$$\| u \|^2_{2,G} \leq C_1 \cdot \|L u\|^0_{2,G} \quad (u \in H^2_2(G)).$$

There is a partition of unity $\zeta_1, \ldots, \zeta_S$, each $\zeta_s \in C^1_1(\bar{G})$ and having support in one such neighborhood

Now, suppose there is no such constant C in (3.3.3). Then there exists a sequence $\{L_n\}$ of operators and $\{u_n\}$ of functions in $H^2_2(G)$ which vanish along ∂G such that the coefficients of the L_n satisfy our conditions uniformly ($a_n^{\alpha\beta}$ being equicontinuous) such that $\|u_n\|^2_2 = 1$, $L_n u_n \longrightarrow 0$ in $L_2(G)$, and $u_n \longrightarrow 0$ in $L_1(G)$. By choosing a subsequence, we may assume that $u_n \rightharpoonup u = 0$ in $H^2_2(G)$, so that $u_n \longrightarrow u$ in $H^1_{20}(G)$. Then

$$L_n(\zeta_s u_n) = \zeta_s L_n(u_n) + M_s(u_n) \longrightarrow 0 \text{ in } L_2(G)$$

for each s. But then $\|\zeta_s u_n\|^2_2 \longrightarrow 0$ for each s, so that $\|u_n\|^2_2 \longrightarrow 0$, which contradicts our assumption that $\|u_n\|^2_2 = 1$,

The following theorem and its method of proof is essentially due to Nirenberg who explained it to the writer in an informal conversation at the International Congress in 1958.

THEOREM 3.3.2: <u>Under our general hypotheses, there is a real number</u> λ_0 <u>and a constant</u> C, <u>which depend only on the quantities stated in Theorem</u> 3.3.1,

such that

$$\| u \|_2^2 \leq C \, \|Lu + \lambda u\|^0 \quad \text{if } \lambda \text{ real}, \lambda \geq \lambda_0$$

for all u in $H_2^2(G)$ which vanish on G .

Proof. For any given $\eta_1 > 0$, each point x_0 is in a neighborhood or boundary neighborhood in which $|a(x) - a(x_0)| < \eta_1$. We choose a sequence ζ_s, $s = 1, \ldots, \delta$, such that each $\zeta_s \in C_1^1(\bar{G})$ and $\zeta_1^2 + \ldots + \zeta_S^2 = 1$. We let $u_s = \zeta_s u$ and note that

$$\zeta_s Lu = Lu_s + M_s u$$

where each M_s is an operator of the first order. Then

$$(3.3.4) \qquad (Lu, u)_2^0 = \sum_{s=1}^{S} (\zeta_s Lu, \zeta_s u)_2^0 = \sum_s (Lu_s, u_s)_2^0 + (M'u, u)_2^0$$

where M' is of the first order. But

$$(3.3.5) \qquad \begin{aligned} (Lu_s, u_s)_2^0 &= -\int_G a_{0s}^{\alpha\beta} u_{s,\alpha\beta} \bar{u}_s dx - \int_G (a^{\alpha\beta} - a_{0s}^{\alpha\beta}) u_{s,\alpha\beta} \bar{u}_s dx \\ &\quad - \int_G (b^\alpha u_{s,\alpha} + cu_s) \bar{u}_s dx , \quad a_{0s}^{\alpha\beta} = a^{\alpha\beta}(x_{0s}) \end{aligned}$$

where x_{0s} is in the small support of ζ_s . Since, for each s ,

$$-\int_G a_{0s}^{\alpha\beta} u_{s,\alpha\beta} \bar{u}_s dx = \int_G a_{0s}^{\alpha\beta} u_{s,\alpha} \bar{u}_{s,\beta} dx \geq 0 ,$$

we see, by summing the real part of (3.3.5) wih respect to s and using the facts that $u_s = \zeta_s u$, etc., that

$$\operatorname{Re} (Lu, u)_2^0 \geq - \operatorname{Re} \int_G [\sum_s \zeta_s^2 (a^{\alpha\beta} - a_{0s}^{\alpha})] u_{,\alpha\beta} \bar{u} dx$$

$$+ \operatorname{Re}(Mu, u)_2^0$$

where M is an operator of the first order. Thus

$$\operatorname{Re} (Lu, u)_2^0 \geq -\eta_1 \|\nabla^2 u\| \cdot \|u\| - C_1(\eta_1) (\|\nabla u\| + \| u \|) \cdot \|u\|$$

$$\geq - 2\eta_1 \|\nabla^2 u\| \cdot \|u\| - C(\eta_1) \cdot \| u \|^2 . \; (\|\varphi\| = \|\varphi\|_2^0)$$

But, now $(\|\varphi\| = \|\varphi\|_2^0)$, for real λ ,

$$\|Lu + \lambda u\|^2 = \|Lu\|^2 + \lambda^2\|u\|^2 + 2\lambda \operatorname{Re}(Lu, u)$$

$$\geq \|Lu\|^2 + \lambda^2\|u\|^2 - 2\eta_1(\|\nabla^2 u\|^2 + \lambda^2\|u\|^2) - 2\lambda C(\eta_1)\|u\|^2$$

$$\geq \|Lu\|^2(1 - 2\eta_1 C_2) + [\lambda^2(1 - 2\eta_1) - 2\lambda C(\eta_1) - 2\,{}_1C_2]\|u\|^2$$

where C_2 is the constant of Theorem 3.3.1. If we first choose η_1 so small that $(1 - 2\eta_1) \geq 1/2$ and $(1 - 2\eta_1 C_2) \geq 1/2$, we may then choose λ_0 so large that $\lambda^2/2 - 2\lambda C(\eta_1) - 2\eta_1 C_2 \geq 1/2$ for $\lambda \geq \lambda_0$. Then

$$\|Lu + \lambda u\|^2 \geq \frac{1}{2}[\|Lu\|^2 + \|u\|^2]$$

from which the theorem follows.

THEOREM 3.3.3: The conclusions of Theorem 3.2.5 hold under the general hypotheses of this section. In fact no real $\lambda \geq \lambda_0$ is an eigenvalue.

Proof: First, suppose $\lambda_1 \geq \lambda_0$. We amay approximate to L by operators L_n whose coefficients satisfy our conditions uniformly with $a_n^{\alpha\beta}$ converging uniformly to $a^{\alpha\beta}$ on $\bar{G}$, each $a_n^{\alpha\beta} \in C_1^0(\bar{G})$. Define $L_{1n} = L_n + \lambda_1 I$ as an operator on $L_2(G)$ with domain all $u \in H_2^2(G)$ which vanish on ∂G . If, for some n, λ, were an eigenvalue for L_n , then L_{1n} would carry some non-zero element into 0 which would contradict Theorem 3.3.2. Thus λ_1 is not an eigenvalue for any n . Hence, if $f \in L_2(G)$, $L_{1n}^{-1}(f) = u_n$ is defined for each n and $\|u_n\|_2^2 \leq C\|f\|_2^0$ for all n . Hence a subsequence, still called $\{u_n\}$, $\longrightarrow u$ in $H_2^2(G)$ and $u = 0$ on ∂G . Then $L_{1n}u_n \longrightarrow L_1 u$ in L_2 (cf. the proof of Theorem 3.2.3), so that $L_1 u = f$.

Now, the equation $Lu + \lambda u = f$ is equivalent to the equation

$$L_0 u + (\lambda - \lambda_0)u = f \quad (L_0 = L + \lambda_0 I)$$

which is, in turn equivalent to

$$u + (\lambda - \lambda_0) L_0^{-1} u = L_0^{-1} f \;!$$

As an operator on $H_2^2 \cap H_{20}^1$, $L_0^{-1} u$ is compact, since weak convergence in H_2^2 implies strong convergence in L_2 . The theorem follows from the Riesz theory of compact linear operators.

3.4. Holder continuity. In this section, we show that if G is of class $C^{2+\mu}$ and the coefficients and $f \in C^{\mu}(\bar{G})$, then the solutions u obtained in the preceding section $\in C^{2+\mu}(\bar{G})$. Our method is to reduce the problem by regular mappings of class $C^{2+\mu}$ of neighborhoods or boundary neighborhoods of each point onto spheres B_{R_0} or hemispheres G_{R_0} , as described in the proof of Theorem 3.3.1, so that $a^{\alpha\beta}(0) = \delta^{\alpha\beta}$, and then prove local differentiability.

To prove the local differentiability, we assume that u is a solution in $H_2^2(B_{R_0})$ of (3.1.1). For $0 < R \leq R_0$, we define

(3.4.1) $$u_R = P_R(-\Delta u) , \; H_R = u - u_R , \; (\text{so } u = u_R + H_R)$$

where $-P_R(f)$ denotes the potential of f as defined by (2.6.10). In case $\nu = 2$, $-P_R(f)$ differs from the potential of f by that constant chosen so that its average is 0; this gets rid of the log term in the bound for $\| u \|_{2,R}^0$ (see Theorem 2.7.3). We regard u as known, whence u_R and H_R are known and H_R is harmonic. We then define the space ${}^*C^{2+\mu}(B_R)$ to consist of all $u \in H_2^2(B_R)$ such that $u \in C^{2+\mu}(B_r)$ for each $r < R$ with norm

(3.4.2) $${}^*\| u \|_R^{2+\mu} = \max \,[\,' \|u\|_{2,R}^2 \,, \; \sup (R - r)^{\tau+\mu} h_\mu(\nabla^2 u, B_r)] \; (\tau = \nu/2)$$

We first note that if $H_R \in H_{2,R}^2$, then it also $\in {}^*C_R^{2+\mu}$. Then, regarding H_R as known, we obtain an operator equation for u_R as follows:

(3.4.3) $$\begin{aligned} u_R &= P_R(-\Delta u) = P_R(Lu) + P_R[(a - a_0) \cdot \nabla^2 u + b \cdot \nabla u + cu] \\ &= T_R(u_R) + V_R \end{aligned}$$

where

$$(3.4.4) \qquad T_R(u_R) = P_R[(a - a_0)\cdot \nabla^2 u_R + b\cdot \nabla u_R + cu_R] ,$$

$$v_R = P_R(f) + P_R[(a - a_0)\cdot \nabla^2 H_R + b\cdot \nabla H_R + cH_R] .$$

in which v_R is known and εH_2^2 or ${}^*C^{2+\mu}$ according as $f \varepsilon L_2$ or to C^μ. It is then shown that if R is small enough, the norm of T_R is $\leq 1/2$ as an operator in either space. Thus, if R is small enough, the equation (3.4.3) has a unique solution u_R in H_2^2 for each v_R in H_2^2 and also one in ${}^*C^{2+\mu}$ for each v_R in ${}^*C^{2+\mu}$. Thus, if $f \varepsilon C^\mu$ and u is any solution of (3.1.1) in $H_2^2(B_R)$, then $u \varepsilon {}^*C^{2+\mu}(B_R)$. A corresponding program is possible for functions u on G_R which vanish along σ_R.

We define the auxiliary space ${}^*C_R^\mu$ to consist of all $f \varepsilon L_2(B_R)$ such that $f \varepsilon C^\mu(B_R)$ for each $r < R$ and for which

$$^*|||f|||_R^\mu = \max [\|f\|_{2,R}^0 , \sup (R - r)^{\tau+\mu} h_\mu(f, B_r)] < \infty .$$

LEMMA 3.4.1: <u>If H is harmonic and $\varepsilon H_2^1(G_R)$ and vanishes along σ_R and if H is extended to the lower hall of B_R by formula (3.2.2), then H is harmonic on B_R and $\varepsilon H_2^1(B_R)$; if $H \varepsilon H_2^2(G_R)$, then $H \varepsilon H_2^2(B_R)$.</u>

<u>Proof</u>: For if $v \varepsilon C_c^1(B_R)$, then

$$\int_{B_R} v_{,\alpha} H_{,\alpha}\, dx = \int_{G_R} w_{,\alpha} H_{,\alpha}\, dx = 0 \quad , \text{where}$$

$$w(x^\nu, x'_\nu) = v(x^\nu, x'_\nu) - v(-x^\nu, x'_\nu)$$

and $w = 0$ on ∂G_R.

THEOREM 3.4.1: <u>If H is harmonic and $\varepsilon H_2^2(B_R)$, then $H \varepsilon {}^*C^{2+\mu}(B_R)$ and</u> ${}^*|||H|||_R^{2+\mu} \leq C(\mu,\nu)' \|H\|_{2,R}^2$.

<u>PROOF</u>. Since $\nabla^2 H$ is harmonic, it follows from Exercise 2, § 1.2, that

$$|\nabla^3 H(x)| \leq C(\nu)\cdot \|\nabla^2 H\|_{2,R}^0 \cdot (R - |x|)^{-\tau-1} \leq C\|H\|_{2,R}^2 \cdot (R - r)^{-\tau-\mu}(r-|x|)^{\mu-1}, \ x \varepsilon B_r$$

from which the result follows as in the proof of Theorem 1.

THEOREM 3.4.2: <u>The transformation</u> $u = P_R(f)$ <u>is a bounded operator from</u> ${}^*C^{\mu}_R$ <u>into</u> ${}^*C^{2+\mu}_R$ <u>with bounded independent of</u> R .

<u>Proof</u>: $P_R(f) \varepsilon H^2_2(B_R)$ with $'\| P_R(f) \|^2_{2,R} \leq C \|f\|^0_{2,R}$ by Theorem 2.6.2 and our definition when $\nu = 2$. For each r with $0 < r < R$, let $r' = (r + R)/2$ and let $f_1(x) = f(x)$ on $B_{r'}$, $f_1(x) = 0$ elsewhere, $f_2(x) = f(x) - f_1(x)$, and let $u_k = P_R(f_k)$ for $k = 1, 2$. Then from the corollary to Theorem 1.5.4, it follows that $u_1 \varepsilon C^{2+\mu}(\bar{B}_{r'})$ with

(3.4.5) $$h_\mu(\nabla^2 u_1, B_r) \leq C_1(\mu, \nu)\cdot h_\mu(f, B_{r'}) \leq C_2\cdot(R-r)^{-\tau-\mu} K, \; K = {}^*|||f|||^{\mu}_R .$$

Since $f_2 = 0$ in $B_{r'}$, u_2 is harmonic there and $u_2 \varepsilon H^2_2(B_{r'})$ with

$$'\| u_2 \|^2_{2,r} \leq C_3 \|f\|^0_{2,R}$$

From Theorem 3.4.1, it follows that $u_2 \varepsilon C^{2+\mu}(B_r)$ with

(3.4.6) $$h_\mu(\nabla^2 u_2, B_r) \leq C_4(\mu, \nu) K \, R-r)^{-\tau-\mu}, \; K = {}^*|||f|||^{\mu}_R .$$

The result follows from (3.4.5) and (3.4.6)

LEMMA 3.4.2: <u>There are constants</u> $C_k(\mu, \nu)$ <u>such that</u>

(a) $|||f|||^0_r \leq C_1 \, {}^*|||f|||^{\mu}_R \cdot (R - r)^{-\tau}$, $0 < r < R$, $f \in {}^*C^{\mu}_R$

(b) $|||\nabla^2 u(x)|||^0_r \leq C_1 K\cdot(R - r)^{-\tau}$, $h_\mu(\nabla u, B_r) \leq C_2 K(R - r)^{1-\tau-\mu}$,

$$|||\nabla u(x)|||^0_r \leq C_3 K R^{\mu}(R - r)^{1-\tau-\mu} , \; h_\mu(u, B_r) \leq C_4 R(R - r)^{1-\tau-\mu} ,$$

$$|||u(x)|||^0_r \leq C_5 K R^{1+\mu}(R - r)^{1-\tau-\mu} , \; 0 < r < R , \; u \varepsilon {}^*C^{2+\mu}_R .$$

<u>Proof</u>: We prove (b); the proof of (a) is similar. Writing

$$\nabla^2 u(x) = \nabla^2 u(\xi) - [\nabla^2 u(\xi) - \nabla^2 u(x)], \; x \varepsilon B_r, \xi \varepsilon B(x, \rho), \rho = (R-r)/2$$

and applying the Schwarz and Minkowski Inequalities over $B(x, \rho)$, we obtain

$$\gamma \nu^{1/2} \rho^{\tau} |\nabla^2 u(x)| \leq \|\nabla^2 u\|_R^0 + K \rho^{-\tau-\mu} \gamma \nu^{1/2} \rho^{\tau} \rho^{\mu}$$

from which the inequality for $\|\nabla^2 u\|_r^0$ follows. Since

$$|\nabla^2 u(x)| \leq C_1 K \cdot (R - |x|)^{-\tau} \leq C_1 K (R - r)^{1-\tau-\mu} (r - |x|)^{\mu-1}, \; x \in B_r,$$

the result for $h_\mu(\nabla u, B_r)$ follows (see the proof of Theorem 1.5.4). The remaining inequalities follow by using similar tricks.

THEOREM 3.4.3: Suppose $a^{\alpha\beta}$, b^α, and $c \in C^\mu(\bar{B}_{R_0})$ with $a^{\alpha\beta}(0) = \delta^{\alpha\beta}$. Then there is a number R_1 with $0 < R_1 < R_0$ such that if $0 < R \leq R_1$, $f \in {}^*C^\mu(B_R)$, $u \in H_2^2(B_R)$ and u is a solution of (3.1.1) on B_R, then $u \in {}^*C^{2+\mu}(B_R)$ and

$$^*\|u\|_R^{2+\mu} \leq C[^*\|Lu\|_R^\mu + {}'\|u\|_{2,R}^2]$$

where C depends on μ, ν, and the bounds and Holder constants of the coefficients.

Proof: As an operator on $H_2^2(B_R)$ and $^*C^{2+\mu}(B_R)$, T_R is easily seen from (3.4.4), Lemma 3.4.2, and Theorem 3.4.2, to have a bound $\leq CR^\mu$ for $R \leq R_0$ and so its bound in either space is $\leq 1/2$ if $0 < R \leq$ some $R_1 \leq R_0$. Thus if $f \in {}^*C^\mu$, $P_R(f) \in {}^*C^{2+\mu}$ with $^*\|P_R f\|^{2+\mu} \leq C_1(\mu, \nu) {}^*\|f\|^\mu$. Moreover $'\|u_R\|_{2,R}^2 \leq {}'\|u\|_{2,R}^2$ so $'\|H_R\|_{2,R}^2 \leq 2' \|u\|_{2,R}^2$ so $H_R \in {}^*C_R^{2+\mu}$ with $^*\|H_R\|^{2+\mu} \leq C_2(\mu, \nu) \cdot {}'\|u\|_{2,R}^2$, so that $v_R \in {}^*C_R^{2+\mu}$ with

$$^*\|v_R\|_R^{2+\mu} \leq C_1^* \|f\|_R^\mu + C_3 R^\mu \, '\|u\|_{2,R}^2$$

where C_3 depends on the quantities stated. If $0 < R \leq R_1$, there is a unique solution u_R in $H_{2,R}^2$ of equation (3.4.3) and also one in $^*C_R^{2+\mu}$, so these must coincide. Thus $u = u_R + H_R \in {}^*C^{2+\mu}(B_R)$ and the inequality holds.

A similar program can be carried through on hemispheres G_R. We define the space $^*C^{2+\mu}(G_R)$ to consist of all $u \in H_2^2(G_R)$ which vanish along σ_R

and have finite norm as defined by (3.4.2) with G_r replacing B_r. We define the space ${}^*C^{\mu}(G_R)$ as before with G_r replacing B_r; we do not require f to be zero along σ_R'. We define

$u = P_R(f)$ for $f \varepsilon G_R$ by

$$-u(x) = \int_{G_R} [K_0(x - \xi) - K_0(x' - \xi)]\, f(\xi)d\,,\quad ((x^{\nu}, x'_{\nu})' = (-x^{\nu}, x'_{\nu}))$$

(3.4.7)

$$= \int_{B_R} K_0(x - \xi)f(\xi)d\xi - 2\int_{G_R^-} K_0(x - \xi)f(\xi)d\xi\,,$$

where f is extended to G_R^- by the formula

(3.4.8) $$f(x^{\nu}, x'_{\nu}) = f(-x^{\nu}, x'_{\nu}) \quad \text{for } x^{\nu} < 0.$$

Then T_R is defined by (3.4.4).

THEOREM 3.4.4: (a) <u>If</u> H <u>is harmonic</u>, $H \varepsilon H_2^2(G_R)$, <u>and</u> H <u>vanishes along</u> σ_R, <u>then</u> $H \varepsilon {}^*C^{2+\mu}(G_R)$ <u>and</u>

$$^*\|H\|_R^{2+\mu} \leq C(\mu, \nu)\, \|H\|_{2,R}^2 .$$

(b) <u>Theorem 3.4.2 holds with the current interpretation.</u>

<u>Proof</u>: Part (a) follows from Lemma 3.4.1 and Theorem 3.4.1. To prove (b), we note from (3.4.7), (3.4.8), and Theorem 2.6.2 that $P_R(u) \varepsilon H_2^2(G_R)$ and u vanishes along σ_R'. If $f \varepsilon {}^*C^{\mu}(G_R)$, then $f \varepsilon {}^*C^{\mu}(B_R)$ so the inequalities hold for it. To handle the integral over G_R^-, we break up $f = f_1 + f_2$ as before and let u_k be the corresponding integral. As before, u_2 is harmonic on $B_{r'}$ and so satisfies the desired inequalities. Finally, if $\alpha < \nu$

$$-u_{1,\alpha\beta\gamma}(x) = -2f(x)\int_{G_{r'}^-} K_{0,\alpha\beta\gamma}(x - \xi)d\xi - 2\int_{G_{r'}^-} K_{0,\alpha\beta\gamma}(x - \xi)[f(\xi)-f(x)]d\xi$$

(3.4.8)

$$= 2f(x)\int_{\Sigma_{n'}^-} K_{0,\beta\gamma}(x - \xi)d\xi'_{\alpha} - 2\int_{G_{r'}^-} K_{0,\alpha\beta\gamma}(x - \xi)[f(\xi)-f(x)]d\xi$$

so that

$$|u_{1,\alpha\beta\gamma}(x)| \leq C_1 K \cdot (R - r)^{-\tau}(r' - |x|)^{-1} + C_2 K(R - r)^{-\tau+\mu} \cdot (x^{\nu})^{\mu-1}$$

$$\leq CK(R - r)^{-\tau-\mu}[d(x, \partial G_r)]^{\mu-1} \;, \quad \text{if} \quad \alpha < \nu.$$

But, since u_1 is harmonic, the same inequality holds for $u_{1,\nu\nu\nu}$. The bound on $h_\mu(\nabla^2 u, G_r)$ follows as in the proof of Theorem 1.

LEMMA 3.4.3: The inequalities of Lemma 3.4.2 hold.

THEOREM 3.4.5: Theorem 3.4.3 holds with B_R replaced by G_R, provided we assume also that $u = 0$ along σ_R.

The proof is the same as that of Theorem 3.4.3.

THEOREM 3.4.6: Suppose G is of class $C^{2+\mu}$, the $a^{\alpha\beta}$, b^α, and $c \in C^\mu(\bar{G})$, and the $a^{\alpha\beta}$ satisfy (3.1.2). Then the conclusions of Theorem 3.3.3 hold. In addition any solution of (3.1.1) in which $f \in C^\mu(\bar{G}) \in C^{2+\mu}(\bar{G})$ whether $\lambda \in \mathcal{E}$ or not. Finally, if $c(x) \leq 0$ on $\bar{G}$, then λ_0 may be taken equal to 0 in Theorem 3.3.3, and there is a constant C, which depends only on G, μ, ν, h, and the bounds and Holder constants of the coefficients, such that

$$\| u \|^2_{2,G} \leq C \| Lu \|^0_{2,G} \;, \quad \|| u \||^{2+\mu}_G \leq C \|| Lu \||^\mu_G$$

for $u \in H^2_2(G)$ and $C^{2+\mu}(\bar{G})$, respectively.

Proof: The proofs of the first two statements have been given. If $u \in H^2_2(\bar{G})$ and $Lu + \lambda u = 0$ with $\lambda \geq 0$, then $u \in C^{2+\mu}(\bar{G})$. But if $c(x) \leq 0$, we conclude from the Maximum Principle (Theorem 1.6.2) that $u = 0$. To see that the constant exists, suppose the contrary. Then there exists a sequence of operators L_n and functions u_n such that $\| u_n \|^2_2 = 1$, $u_n \longrightarrow$ in H^2_2, $c_n(x) \leq 0$ for each n, and the $a^{\alpha\beta}_n \longrightarrow a^{\alpha\beta}$, $b^\alpha_n \longrightarrow b^\alpha$, and $c_n \longrightarrow c$ uniformly on $\bar{G}$, and $L_n u_n \longrightarrow 0$ in L_2. Then, since $L_n u_n \longrightarrow Lu$ in L_2, we see that

$Lu = 0$, so $u = 0$. But then $u_n \longrightarrow u$ in L_2 so $\|u_n\|_2^2 \longrightarrow 0$ on account of Theorem 3.3.1.

Now, suppose $Lu \in C^{\mu}(\bar{G})$. From Theorems 3.4.3 and 3.4.5 and the argument in the first paragraph of the proof of Theorem 3.3.1, it follows that there is a constant C, depending only on the quantities stated, and that each point of $\bar{G}$ is in a neighborhood or boundary neighborhood N such that

$$|||\, u \,|||_N^{2+\mu} \leq C[\, |||\, Lu \,|||_G^{\mu} + \|u\|_{2,G}^2]$$

The second result then follows.

REMARKS: Local differentiability properties analogous to those stated in Theorem 3.2.4 hold.

<u>3.5.</u> <u>Higher differentiability</u>. In this section, we show that additional differentiability of the coefficients and of f near an interior point implies additional differentiability of any solution u of (3.1.1); and if the point is on ∂G and u vanishes along a smooth part of ∂G, then u possesses additional differentiability in a boundary neighborhood. We begin by proving this for spheres or hemispheres and pass to the general case by mappings. The method of proof in these special cases involves boundedness theorems and the difference quotient procedure of § 3.2.

LEMMA 3.5.1: <u>Suppose</u> $u \in H_2^2(B_R)$, <u>or</u> $u \in H_2^2(G_R)$ <u>and vanishes along</u> σ_R. <u>Then there is a constant</u> $C = C(\nu)$ <u>such that</u>

$$\|\nabla u\|_r \leq \varepsilon(R - r)\|\nabla^2 u\|_R + C\,\varepsilon^{-1}(R - r)^{-1}\|u\|_R,\ 0 < \varepsilon \leq 1,$$

<u>where</u> $\|\varphi\|_r = \|\varphi\|_2^0$ <u>on</u> B_r <u>or</u> G_r <u>as the case may be.</u>

<u>Proof</u>: Let $\zeta(x) = h[(|x| - r)/(R - r)]$. Then

$$\int_{G_R} \zeta^2 |\nabla u|^2 dx = \int_{G_R} \bar{u}_{,\alpha}(\zeta^2 u_{,\alpha})dx = -\int_{G_R} \bar{u}\cdot(\zeta^2 \Delta u + 2\zeta\zeta_{,\alpha} u_{,\alpha})\,dx$$

$$\leq \nu^{1/2}\| u\|_R \cdot \|\nabla^2 u\|_R + 2h_1(R - r)^{-1}\|u\|_R \cdot \|\zeta\nabla u\|_R$$

$$\leq \eta(R - r)^2\|\nabla^2 u\|_R^2 + \nu\cdot(4\eta)^{-1}(R - r)^{-2}\|u\|_R^2 + \tfrac{1}{2}\|\zeta\nabla u\|_R^2$$

$$+ C_1(R - r)^{-2}\| u \|_R^2$$

for any $\eta > 0$. The result follows by transposing the term $\frac{1}{2}\|\zeta\nabla u\|_R^2$ to the left side and noticing that

$$\|\nabla u\|_r \leq \|\zeta\nabla u\|_R .$$

LEMMA 3.5.2: <u>Suppose that</u> $u \varepsilon H_2^2(B_R)$ <u>with</u> $\Lambda(u) \subset B_R$, <u>or</u> $u \varepsilon H_2^2(G_R)$ <u>with</u> $\Lambda(u) \subset B_R$ <u>and</u> u <u>vanishes along</u> σ_R. <u>Then</u>

(3.5.1) $$\|\nabla^2 u\|_2^0 = \| -\Delta u\|_2^0$$

<u>Proof</u>: We first prove this for B_R. Let $\tilde{u}$ be the Fourier transform of u:

$$\tilde{u}(y) = (2\pi)^{-\nu/2} \int_{B_R} e^{ix\cdot y} u(x)dx$$

Then (3.5.1) follows from the Plancherel theorem. The theorem follows for G_R by reflecting u across σ_R by formula (3.3.2).

THEOREM 3.5.1: <u>Suppose the coefficients</u> $a^{\alpha\beta}$, b^α <u>and</u> c <u>satisfy the hypotheses of section</u> 3.3 <u>on</u> B_{R_0} <u>with</u> $a^{\alpha\beta}(0) = \delta^{\alpha\beta}$. <u>Then there is an</u> R_1 <u>with</u> $0 < R_1 \leq R_0$, <u>which depends only on the bounds for the</u> b^α <u>and</u> c <u>and the moduli of continuity of the</u> $a^{\alpha\beta}$, <u>such that if</u> u <u>satisfies the hypotheses of Lemma</u> 3.5.2, <u>then</u>

$$\|\nabla^2 u\|_2^0 \leq 2\| Lu\|_2^0$$

<u>Proof</u>: For, from Lemma 3.5.2, we conclude that

$$\|\nabla^2 u\|_2^0 = \| -\Delta u\|_2^0 \leq \| Lu\|_2^0 + \|(a^{\alpha\beta} - a_0^{\alpha\beta})u_{,\alpha\beta} + b^\alpha u_{,\alpha} + cu \|_2^0$$

$$\leq \| Lu\|_2^0 + [\varepsilon(r) + BR + CR^2] \|\nabla^2 u \|_2^0 , \lim_{R\to 0} \varepsilon(R) = 0$$

Clearly $\varepsilon(R) + BR + CR^2 \leq 1/2$ if R is small.

THEOREM 3.5.2: <u>Suppose the coefficients satisfy the conditions of Theorem 3.5.1 and</u> $0 < R \leq R_1$. <u>Suppose that</u> (a) $u \varepsilon L_2(B_R)$ <u>and</u> $u \varepsilon H_2^2(B_r)$ <u>for each</u> $r < R$, <u>or</u> (b) $u \varepsilon L_2(G_R)$ <u>and</u> $u \varepsilon H_2^2(G_r)$ <u>for each</u> $r < R$ <u>and vanishes along</u> σ_R . <u>Then there is a</u> $C = C(\mathcal{V})$ <u>such that</u>

(3.5.2) $$\|\nabla^2 u\|_r \leq K(R - r)^{-2} , \quad \|\nabla u\|_r \leq K(R - r)^{-1}, \text{ where}$$

$$K = C \cdot [R^2 \| Lu \|_R + \| u \|_R]$$

<u>Proof</u>: It is sufficient to prove this for $r \geq R/2$. First choose $r^1, r < r^1 < R$, and define

$$\zeta(x) = h[(|x| - r)/(r' - r)] , \quad U = \zeta u$$

Then U satisfies the hypotheses of Theorem 3.5.1 and

(3.5.3) $$LU = \zeta Lu - 2a^{\alpha\beta} \zeta_{,\beta} u_{,\alpha} - (a^{\alpha\beta} \zeta_{,\alpha\beta} + b^{\alpha}\zeta_{,\alpha}) u$$

From Theorem 3.5.2 (and the fact that $r' \leq R_1 \leq R_0$ and $r \geq r^{1}/2)$ we conclude that

(3.5.4) $$\|\nabla^2 u\|_r \leq \|\nabla^2 U\|_{r'} \leq 2[\| Lu\|_{r'} + 3h_1 \|\nabla u\|_{r'} \cdot (r' - r)^{-1} + \tfrac{3}{2}\zeta_2 \|u\|_{r'} \cdot (r'-r)^{-2}]$$

$$\zeta_2 = h_2 + h_1 \sqrt{2}$$

To prove (3.5.2), let R' be any number $r < R' < R$, let $r' = r + \delta$, $r'' = r + 2\delta$, $R' = r + 3\delta$ and let

$$K_1 = \max_{0 \leq r \leq R'} (R' - r)^2 \|\nabla^2 u\|_r .$$

Then from (3.5.4) and Lemma 3.5.1, we conclude that

$$\|\nabla^2 u\|_r \leq 2[\| Lu\|_{r'} + 3h_1 \varepsilon \|\nabla^2 u \|_{r''} + (\tfrac{3}{2} \zeta_2 + 3h_1 C \varepsilon^{-1}) \| u \|_{r''} \delta^{-2}]$$

If we choose ε so that $54h_1 \varepsilon \leq 1/2$, we conclude that

$$(R' - r)^2 \|\nabla^2 u\|_r \leq \frac{1}{2} K_1 + 2\| Lu \|_{r'} R^2 + 18(\frac{3}{2} \zeta_2 + 3h_1 C t^{-1}) \|u\|_{r''}$$

Thus we obtain

$$\|\nabla^2 u\|_r \leq K_1 \cdot (R' - r)^{-2}, \quad K_1 = C_1 [\| Lu \|_R \cdot R^2 + \| u \|_R]$$

From Lemma 3.5.1, we conclude that (3.5.2) holds for R' with a fixed C independent of R'. We may then allow $R' \to R$.

THEOREM 3.5.3: Suppose the coefficients and f satisfy the conditions above on B_{R_0} or G_{R_0} and, in addition, suppose $0 < R \leq R_1 \leq R_0$ and that $a^{\alpha\beta}$, b^α, and $c \in C_1^0(B_R)$ or $C_1^0(G_R)$ and $f \in H_2^1(B_R)$ or $H_2^1(G_R)$ as required below. Suppose that (a) $u \in H_2^2(B_R)$ or (b) $u \in H_2^2(G_R)$ and vanishes along σ_R and satisfies (3.1.1) with $\lambda = 0$ in either case. Then $u \in H_2^3(B_r)$ in (a) or $H_2^3(G_r)$ in (b) for each $r < R$ and the derivatives $u_{,\gamma}$ satisfy the differentiated equations.

Proof: We first prove this for B_R. Suppose $0 < r < R$, $3\delta = R - r$, $r' = r + \delta$, $r'' = r + 2\delta$. Suppose $1 \leq \gamma \leq \nu$, $0 < |h| < \delta$, e_γ is the unit vector in the x^γ direction, and we define

$$u_h(x) = h^{-1}[u(x + he_\gamma) - u(x)]$$

with similar definitions for $a_h^{\alpha\beta}$, etc. Forming the corresponding difference quotient of equation (3.1.1) with $\lambda = 0$, we obtain

$$Lu_h(x) = F_h(x), \quad \text{where}$$

(3.5.5)

$$F_h(x) = f_h(x) + a_h^{\alpha\beta}(x)\, u_{,\alpha\beta}(x + he_\gamma) + b_h^\alpha(x) u_{,\alpha}(x + he_\gamma) + c_h(x) u(x+he_\gamma)$$

From (3.5.5), Lemma 2.4.1, etc., we see that

$$(3.5.6) \qquad u_h \to u_{,\gamma}, \quad F_h \to F = f + a^{\alpha\beta}_{,\gamma} u_{,\alpha\beta} + b^\alpha_{,\gamma} u_{,\alpha} + c_{,\gamma} u \quad \text{in } L_2(B_{r''}).$$

Accordingly, we see from Theorem 3.5.2 that the u_h form a Cauchy family

in $H_2^2(B_r)$ which, from (3.5.6), must have $u_{,\gamma}$ as a limit. This proves the theorem for B_r.

In the case of G_R, the procedure above applies for each $\gamma \leq \nu - 1$ over the whole of G_R and for $\gamma = \nu$ on any part of G_R where $y^\nu \geq \varepsilon > 0$. From this, we conclude that all the third derivatives except $u_{,\nu\nu\nu} \in L_2(G_r)$ for $r < R$ and that $u_{,\nu}$ satisfies the differentiated equation on the interior of G_R. Since $a^{\nu\nu}$ is nearly 1, we can solve for $u_{,\nu\nu\nu}$ in terms of the other derivatives, we see that it also $\in L_2(G_r)$ so that $u_{,\nu}$ also $\in H_2^2(G_r)$ for each $r < R$.

THEOREM 3.5.4: <u>If, in Theorem</u> 3.5.3, <u>the coefficients and</u> $f \in C_\mu^{\prime}(\bar{B}_R)$, or $C_\mu^{\prime}(\bar{G}_R)$, <u>then</u> $u \in C_\mu^3(B_r)$ <u>or</u> $C_\mu^3(G_r)$, <u>respectively, for</u> $r < R$.

<u>Proof</u>: For, by Theorem 3.4.3, we see first that $u \in C_\mu^2(B_{r''})$ or $C_\mu^2(G_{r''})$, using the notation of the proof above. Applying the difference quotient procedure, we see that F_h converges uniformly on $B_{r'}$ (or $G_{r'}$) to F as defined in (3.5.6) with $\|F_h\|_\mu^0$ uniformly bounded. Then $u_{,\gamma}$ satisfies an equation $Lu_{,\gamma} = F$ where $F \in C_\mu^0$, so that $u_{,\gamma} \in C_\mu^2(B_r)$. The case for G_R is similar.

It is clear that the processes above may be repeated and it is also clear what types of results hold for solutions of (3.1.1) on smooth domains. For example, we state:

<u>Suppose the coefficients and</u> $f \in C_\mu^k(\bar{G})$ <u>where</u> G <u>is of class</u> C_μ^{k+2}, $0 < \mu < 1$, <u>and suppose</u> $u \in H_2^2(G)$ $H_{20}^{\prime}(G)$ <u>and satisfies</u> (3.1.1) <u>almost everywhere on</u> G <u>for some</u> λ. <u>Then</u> $u \in C_\mu^{k+2}(\bar{G})$ <u>and</u> u <u>vanishes on</u> ∂G <u>in the ordinary sense</u>.

<u>3.6. Lower-order differentiability</u>. Developments corresponding to those

in § 3.4 can be carried out for the equations (3.1.4). The idea is to restrict consideration to the case where

$$a^{\alpha\beta}(0) = \delta^{\alpha\beta} \tag{3.6.1}$$

and to write

$$u = u_R + H_R \tag{3.6.2}$$

where H_R is harmonic. Then u_R must satisfy the equation

$$\int_G \{v_{,\alpha}[\delta^{\alpha\beta} u_{R,\beta} + (a^{\alpha\beta} - \delta^{\alpha\beta})u_{,\beta} + b^\alpha u + e^\alpha] + v(c^\alpha u_{,\alpha} + du + f)\} dx = 0, \tag{3.6.3}$$

for all $v \in H^1_{20}(G)$, where $G = B_R$ or G_R. In the case of B_R, we take

$$u_R = Q_R[(a - a_0)\cdot\nabla u + bu + e] + P_R[c \cdot \nabla u + du + f] \tag{3.6.4}$$

where we $Q_R(E)$ denotes the quasi-potential of the vector of E as defined in (2.6.1) and $P_R(F)$ is the potential of F as modified before in case $\nu = 2$. Then, as before,

$$u_R - T_R u_R = v_R \tag{3.6.5}$$

where

$$T_R u_R = Q_R[(a - a_0) \cdot \nabla u_R + bu_R] + P_R [c \cdot \nabla u_R + du_R]$$

$$v_R = Q_R[(a - a_0) \cdot \nabla H_R + bH_R + e] + P_R[c \cdot \nabla H_R + dH_R + f] .$$

Then it is desirable to introduce the spaces ${}^*C^{1+\mu}(B_R)$, ${}^*C^\mu(B_R)$, and ${}^*M^\mu(B_R)$ with norms (for ${}'\| u \|^1_2$, see Theorem 2.5.4)

$$
\begin{aligned}
&{}^*|||u|||^{1+\mu}_R = \max [{}'\|u\|^1_{2,R}, \ \sup_r \ (R - r)^{\tau+\mu} h_\mu(\nabla u, B_r)] , \\
&{}^*|||e|||^\mu_R = \max [\| e \|^0_{2,R}, \ \sup_r \ (R - r)^{\tau+\mu} h_\mu(e, B_r)] , \qquad (3.6.7) \\
&{}^{*\prime}||| f |||^\mu_R = \max [\| f \|^0_{2,R}, \ \sup_r R^{-\mu}(R - r)^{\gamma+\mu} ||| f|||^0_r] ,
\end{aligned}
$$

where the required continuity is implied except in the case of ${}^*M^\mu$ where f is merely bounded and measurable on each B_r and $|||f|||^0_r$ is its essential

sup. there. Then one can prove the lemmas and theorem:

LEMMA 3.6.1: (a) Q_R is a bounded operator from ${}^*C^\mu(B_R)$ to ${}^*C^{1+\mu}(B_R)$ with bound independent of R.

(b) P_R is a bounded operator from ${}^*M^\mu$ to ${}^*C^{1+\mu}$ with a bound of the form $C(\mu,\nu)\cdot R$.

LEMMA 3.6.2: There are constants $C_k(\nu)$ such that

(a) $\|\!| e \|\!|_r^0 \le C_1 K_1 \cdot (R - r)^{-\tau}$, $0 < r < R$, $e \varepsilon {}^*C^\mu$, $K_1 = {}^*\|\!| e \|\!|^\mu$;

(b) $\|\!| \nabla u \|\!|_r^0 \le C_1 K_2 (R - r)^{-\tau}$, $h_\mu(u, B_r) \le C_2 K_2 (R - r)^{1-\mu-\tau}$,

(c) $\|\!| u \|\!|_r^0 \le C_3 K_2 R^\mu (R - r)^{1-\tau-\mu}$, $u \varepsilon {}^*C^{1+\mu}$, $K_2 = {}^*\|\!| u \|\!|^{1+\mu}$, $\tau = \nu/2$.

THEOREM 3.6.1: If H is harmonic and $\varepsilon H_2^1(B_R)$, then $H \varepsilon {}^*C^{1+\mu}(B_R)$ and

$$ {}^*\|\!| H \|\!|_R^{1+\mu} \le C(\mu,\nu)' \| H \|_{2,R}^1 $$

Note also Theorem 3.4.1.

THEOREM 3.6.2: Suppose the coefficients satisfy the conditions of § 3.1 on $\bar{B}_{R_0}$ and, in addition that

$$ |a(x_2) - a(x_1)| \le A_1 |x_2 - x_1|^\mu, \quad |b(x_2) - b(x_1)| \le B_1 |x_2 - x_1|^\mu, \quad 0 < \mu < 1 $$

on $\bar{B}_{R_0}$. Then there is an R_2 with $0 < R_2 \le R_0$, which depends only on μ, ν, and the bounds and Holder constants for the coefficients, such that ${}^*\|\!| T_R \|\!|^{1+\mu} \le 1/2$ if $0 < R \le R_2$.

For the case of G_R, the corresponding program can be carried through; it is necessary to restrict attention to functions u which vanish along σ_R. The spaces are defined as above with G_r replacing B_r, $u = 0$ along σ_R in the case ${}^*C^{1+\mu}$. Then, for e and $f \varepsilon L_2(G_R)$, we define $u = Q_R(e)$ and $v = P_R(f)$ by the formulas

$$u(x) = \int_{B_R} K_{0,\alpha}(x - \xi) e^{\alpha}(\xi) d \quad - \quad 2 \sum_{\alpha=1}^{\nu=1} \int_{G_R^-} K_0(x - \xi) e(\xi) d\xi$$

(3.6.8)

$$v(x) = \int_{B_R} K_0(x - \xi) f(\xi)\, d\xi \quad - \quad 2 \int_{G_R^-} K_0(x - \xi) f(\xi)\, d\xi$$

where the e^{α} and f have been extended to G_R^- by the formulas

$$e^{\alpha}(x^{\nu}, x'_{\nu}) = e^{\alpha}(-x^{\nu}, x'_{\nu}),\ \alpha = 1, \ldots, \nu,\ f(x^{\nu}, x'_{\nu}) = f(-x^{\nu}, x'_{\nu})$$

Then the reader can prove the corresponding lemmas 3.6.1$'$, etc., for the functions on G_R .

EXERCISES

1. Prove Lemma 3.6.1 (a).
2. Prove Lemma 3.6.1 (b).
3. Prove Lemma 3.6.2 (a).
4. Prove Lemma 3.6.2 (b).
5. Prove Theorem 3.6.1.
6. Prove Theorem 3.6.2.
7. Prove Lemma 3.6.1$'$ (a); note that $Q_R(e)$ must $= 0$ on σ_R .
8. Prove Lemma 3.6.1$'$ (b).
9. Prove Theorem 3.6.2$'$.

CHAPTER 4

MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

4.1. Introduction. Interest in the calculus of variations was aroused by various problems in mechanics and geometry some of which were very old. Interest was heightened by Hamilton's Principle in mechanics. The solution of these problems involved the determination of that function (or vector function) z which minimizes (or at least gives a stationary value to) an integral

$$I(z) = \int_a^b f[x, z(x), z'(x)]\,dx \tag{4.1.1}$$

where x denotes a single variable. Riemann aroused interest in a corresponding type of integral involving two independent variables when he discovered many interesting results in function theory by assuming that the so-called Dirichlet integral (see § 2.2) had a minimum with given boundary values. Unfortunately, this matter was not cleared up until about 1900 by Hilbert who proved that this was true under certain conditions.

Riemann's idea was based on the fact that the Euler equation (see below) for the Dirichlet integral is just Laplace's equation. The Euler equation for a solution of a problem

$$\iint_G f[x, y, z(s, y)z_x(x, y)z_y(x, y)]\,dxdy = \min. \tag{4.1.2}$$

is derived as follows: Let us assume that f is of class C'', G is of class C', and the minimizing function z is of class C''. Let $\zeta(x, y)$ be of class C' and vanish on ∂G. Then, for each λ, $z + \lambda\zeta$ has the given boundary values. Since z is minimizing, the function

$$\varphi(\lambda) = \iint_G f(x, y, z + \lambda\zeta, z_x + \lambda\zeta_x, z_y + \lambda\zeta_y)\,dxdy \tag{4.1.3}$$

must have a minimum at $\lambda = 0$. Thus we must have

$$\varphi'(0) = 0 = \iint_G (f_z\zeta + f_p\zeta_x + f_q\zeta_y)\,dx\,dy\ , \tag{4.1.4}$$

where

$$f = f(x, y, z, p, q),\ f_z \text{ means } f_z[x, y, z(x, y), z_x(x, y), z_y(x, y)], \text{ etc.}$$

Since f and z are of class C', we may apply Green's theorem to obtain

$$\iint_G \zeta\left[f_z - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q\right] dxdy = 0 \tag{4.1.5}$$

for every ζ of class C'' which vanishes on ∂G. By approximations, (4.1.5) holds for all ζ in $L_1(G)$ which do not have to vanish on ∂G. It follows that z must satisfy

$$\frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q = f_z \tag{4.1.6}$$

which reduces to Laplace's equation if $f = p^2 + q^2$. It is clear how to derive Euler's equation corresponding to an integral involving more (or fewer) variables x and more variables z.

This derivation requires (1) that there is a minimizing function and (2) that it be of class C'' on $\bar{G}$. That there may not always be a minimizing function is seen by the following example for one function z of one variable x:

$$I(z) = \int_0^1 (1 + z'^2)^{1/4}\, dx,\ z(0) = 0,\ z(1) = 1 .$$

Since the integrand > 1 for every z of class C' with the given boundary values, $I(z) > 1$ for all admitted z. But suppose we define

$$z(x) = \begin{cases} 0 & , 0 \le x \le r \\ -1 + [1 + 3(x - r)^2/(1 - r)^2]^{1/2} & , r \le x \le 1 \end{cases} ,\quad 0 < r < 1 .$$

For this function z,

$$I(z) \le \int_r^1 [1 + 9/(1 - r)^2]^{1/4} dx + r \le r + 10^{1/4}(1 - r)^{1/2}$$

which can be made arbitrarily close to 1 by taking r close enough to 1.

Another difficulty which arises in the Dirichlet problem is the fact that

there can be continuous boundary values for which the Dirichlet integral cannot be finite. From what we now know, the harmonic function minimizes the Dirichlet integral whenever that is finite. Any harmonic function on $B(0, 1)$ can be represented by a series

$$(4.1.7) \qquad u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

which converges absolutely-uniformly with all its derivative series for $r < 1$. Its Dirichlet integral over $B(0, 1)$ is found to be

$$(4.1.8) \qquad D(u, B_1) = \pi \sum_{n=1}^{\infty} n(a_n^2 + b_n^2)$$

If we set

$$\varphi(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} n^{-2} \cos(n!\theta)$$

we see that φ is continuous but if $u(1, \theta) = \varphi(\theta)$ and u is harmonic

$$D(u, B_1) = \pi \sum_{n=1}^{\infty} n^{-4} \cdot n!$$

which diverges.

We now have a condition on f which is <u>necessary</u> if f is to have a minimizing function of class C': Let us assume that f is of class C'' and z is of class C' and minimizes $I(z)$ among functions with the same boundary values. Then if $\varphi(\lambda)$ is defined by (4.1.3), we must have

$$\varphi''(0) = \iint_G [a(x, y)\zeta_x^2 + 2b\zeta_x\zeta_y + c\zeta_y^2 + 2d\zeta\zeta_x + 2e\zeta\zeta_y + \zeta^2]\,dx\,dy \geq 0,$$

(4.1.9)

$$a = f_{pp}[x, y, z(x, y)z_x(x, y), z_y(x, y)], \quad b = f_{pq}, \quad c = f_{qq}, \text{ etc.},$$

for every ζ of class C' which vanishes on ∂G. By approximations, this must hold for all Lipschitz ζ which vanish on ∂G. We note that the coefficients $a, b, \ldots,$ are continuous. Now, let $(x_0, y_0) \in G$ and choose (ξ, η) axes with origin at $P_0(x_0, y_0)$ obtained by rotating through an angle

θ and let $\lambda = \cos\theta$, $\mu = \sin\theta$. Choose a rectangle R with short side $2h \parallel \xi$ axis and long side $2H \parallel y$ axis and center at P_0 and which lies in G. Divide this into triangles by its diagonals and let ζ be the unique function which is continuous on $\bar{G}$, 0 outside R, linear on each triangle of R, and equal to h at P_0. If we form $\varphi''(0)$ for that ζ, choose sequences $\{h_n\}$ and $\{H_n\} \longrightarrow 0$ so that $(h_n/H_n) \longrightarrow 0$, divide by $(h_n H_n/2)$ and pass to the limit, we obtain the result that

(4.1.10) $$f_{pp}\lambda^2 + 2f_{pq}\lambda\mu + f_{qq}\mu^2 \geq 0, \quad f_{pp} = a(x_0, y_0), \text{ etc.},$$

for all λ, μ. If this were to hold for all (x, y, z, p, q), it would follow that f would be convex in (p, q) for each (x, y, z).

We shall consider, in this chapter, integrals of the form

(4.1.11) $$I(z, G) = \int_G f(x, z, \nabla z)dx,$$

where $f(x, z, p)$ is continuous in its arguments

$$x = (x^1, \ldots, x^\nu), \; z = (z^1, \ldots, z^N), \; p = \{p^i_\alpha\}, \; i = 1, \ldots, N; \; \alpha = 1, \ldots, \nu.$$

If we assume that f is of class C'' and repeat the argument of the preceding section (modified properly) we conclude that, if the vector z is of class C' in G and minimizes $I(z)$,

(4.1.12) $$f_{p^i_\alpha p^j_\beta}[x_0, z(x_0), \nabla x(x_0)]\lambda_\alpha\lambda_\beta\xi^i\xi^j \geq 0, \quad x_0 \in G$$

for all vectors $\lambda = (\lambda_1, \ldots, \lambda_\nu)$ and $\xi = (\xi^1, \ldots, \xi^N)$. If $N = 1$, this becomes

$$f_{p_\alpha p_\beta}\lambda_\alpha\lambda_\beta \geq 0$$

which implies that f is convex in p for each (x, z) if it holds for all (x, z, p). The condition (4.1.12) does not imply convexity in all the variables p^i_α taken together and not much is known about integrals subject only to that condition.

DEFINITION: A variational problem in which f satisfies (4.1.12) with the equality excluded for all $\lambda \neq 0$, $\xi \neq 0$, and all (x, z, p) is called <u>regular</u>.

REMARK: We shall see that the Euler equations for a regular variational problem are elliptic.

<u>4.2. Some lower-semicontinuity and existence theorems.</u> In this section we consider integrals (4.1.11) in which $f(x, z, p)$ is convex in all the variables p, G is bounded and of class C', and

(4.2.1) $$f(x, z, p) \geq f_0(p), \quad \lim_{|p| \to \infty} f_0(p)/|p| = +\infty,$$

for being convex.

DEFINITIONS: A set S in a linear space is said to be <u>convex</u> if and only if the segment P_1P_2 belongs to S whenever the points P_1 and P_2 do. A function $\varphi(\xi)$ $(\xi = (\xi^1, \ldots, \xi^P))$ is said to be <u>convex</u> on the convex set S in the ξ-space if and only if

$$\varphi[(1-\lambda)\xi_1 + \lambda\xi_2] \leq (1-\lambda)\varphi(\xi_1) + \lambda\varphi(\xi_2), \quad 0 \leq \lambda \leq 1,$$

whenever ξ_1 and $\xi_2 \in S$.

The following theorems concerning convex functions are well known and are stated without proof:

LEMMA 4.2.1: <u>Suppose $\varphi(\xi)$ is convex on the open convex set S with $|\varphi(\xi)| \leq M$ there. Then φ satisfies</u>

$$|\varphi(\xi_2) - \varphi(\xi_1)| \leq 2M \cdot |\xi_2 - \xi_1|/\delta$$

<u>on any compact subset of S at a distance $> \delta$ from ∂S.</u>

LEMMA 4.2.2: <u>Suppose φ and each φ_n are convex on the open convex set S and suppose $\varphi_n(\xi) \to \varphi(\xi)$ for each ξ on S. Then the convergence is uniform on any compact subset of S.</u>

LEMMA 4.2.3: A necessary and sufficient condition that φ be convex on the open convex set S is that for each $\bar{\xi}$ in S there exists a linear function $a_p \xi^p + b$ such that

$$(4.2.2) \qquad \varphi(\bar{\xi}) = a_p \bar{\xi}^p + b, \quad \varphi(\xi) \geq a_p \xi^p + b \quad \text{for all } \xi \varepsilon S .$$

If φ is of class C' on S, this condition is equivalent to

$$E(\xi, \bar{\xi}) = \varphi(\xi) - \varphi(\bar{\xi}) - (\xi^\alpha - \bar{\xi}^\alpha)\varphi_{,\alpha}(\bar{\xi}) \geq 0, \ \xi, \bar{\xi} \ \varepsilon \ S .$$

If φ is of class C'' on S, this condition is equivalent to

$$\varphi_{,\alpha\beta}(\xi)\eta^{\alpha\beta} \geq 0$$

for all ξ on S and η.

DEFINITION: A linear function $a_p \xi^p + b$ which satisfies (4.2.2) for some ξ is said to be supporting to φ at $\bar{\xi}$.

LEMMA 4.2.4: Suppose φ is convex for all ξ and satisfies

$$(4.2.3) \qquad \lim_{|\xi| \to \infty} \varphi(\xi)/|\xi| = +\infty .$$

Then φ takes on its minimum. Also, if $a_1, \ldots, a_p$ are any numbers, there is a unique b such that $a_p \xi^p + b$ is supporting to φ for some ξ. If ψ is convex and satisfies (4.2.3), if $\psi(\xi) \geq \varphi(\xi)$ for each , and if $a_p \xi^p + c$ is supporting to ψ, then $c \geq b$.

LEMMA 4.2.5: Suppose that φ_n and φ are everywhere convex and satisfy (4.2.3) and suppose that $\varphi_n(\xi) \to \varphi(\xi)$ for each ξ. Suppose $a_1, \ldots, a_P$ are any numbers and b_n and b are chosen so that $a_p \xi^p + b_n$ and $a_p \xi^p + b$ are supporting to φ_n and φ, respectively. Then $b_n \to b$. Likewise, if $a_{np} \to a_p$ for each p and b_n are chosen so that $a_{np} \xi^p + b_n$ and $a_p \xi^p + b$ are all supporting to f, then $b_n \to b$.

The proof of the following lemma is much like that of Lemma 2.4.2.

LEMMA 4.2.6: Suppose $z \in H_1^1(G)$, $\bar{D} \subset G_{\rho_0}$, $0 < \rho < \rho_0$, and φ is a non-negative Friedrichs mollifier. Then

$$(4.2.4) \qquad \int_D [\int_{B(x,\rho)} \rho^{-\nu} \varphi(\frac{\xi - x}{\rho}) |z(\xi) - z(x)| d\xi] dx \le \rho \int_G |\nabla z(y)| dy$$

Proof: Clearly we may choose a domain $D' \subset\subset G$ such that $\bar{D} \subset D'_\rho$ and we may approximate on $\bar{D}'$ to z strongly in $H_1^1(D')$ by functions of class C'. Thus we may assume $z \in C'(D')$. Then

$$|z(\xi) - z(x)| = |(\xi^\alpha - x^\alpha) \cdot \int_0^1 z_{,\alpha}[x + t)(\xi - x)] | dt \le |\xi - x| \cdot \int_0^1 |\nabla z[x + t(\xi - x)]| dt$$

Substituting this in the left side of (4.2.4) which we call I, letting $\eta = \xi - x$, and interchanging the order of integration, we obtain (since $|\eta| \le \rho$)

$$(4.2.5) \qquad I \le \rho \int_{B(0,\rho)} \rho^{-\nu} \varphi(\rho^{-1}\eta) \{\int_0^1 [\int_D |\nabla z(x + t\eta)| dx] dt\} d\eta$$

The result follows by letting $y = x + t\eta$ and noting that the corresponding domain $D(\eta, t) \subset D'$ for each (η, t).

LEMMA 4.2.7 (Jensen's Inequality): If $\varphi(\xi^1, \ldots, \xi^P)$ is convex for all ξ, S is a set, μ is a non-negative bounded measure over S, and the functions $\xi^P \in L_1(S, \mu)$, then

$$\varphi(\bar{\xi}^1, \ldots, \bar{\xi}^P) \le [u(S)]^{-1} \int_S \varphi[\xi^1, \ldots, \xi^P] d\mu, \quad \bar{\xi}^P = [\mu(S)]^{-1} \int_S \xi^P d\mu .$$

Proof: Choose a_p so that

$$\varphi(\bar{\xi}) + a_p(\xi^p - \bar{\xi}^p) \le \varphi(\xi) \quad \text{for all}$$

and then average over S.

LEMMA 4.2.8: Suppose that $f(x, z, p)$ is convex in p for each (x, z), is bounded below, and satisfies a uniform Lepschitz condition for all (x,z,p). Then $I(z)$ is lower semicontinuous with respect to weak convergence in $L_1(G)$, G being bounded.

Proof: From the hypotheses, it follows that

$$I(z) \leq f(0, 0, 0)|G| + \int_G K|x|dx + K\|z\|^1_{1,G} ,$$

K being the Lipschitz constant. Let $D \subset\subset G$ with $\overline{D} \subset G$, suppose $\overline{D}$ G_{ρ_0} , $0 < \rho < \rho_0$, and φ is a non-negative Friedrichs mollifier. Then from Jensen's Inequality with density $\rho^{-\nu}\varphi[(\xi - x)/\rho]$ on $B(x, \rho)$, we conclude that

$$f[x, z_\rho(x), \nabla z_\rho(x)] \leq \int_{B(x,\rho)} f[x, z_\rho(x), \nabla z(\xi)] \rho^{-\nu} \varphi[(\xi - x)/\rho] d\xi$$

$$= F_\rho(x) - \int_{B(x,\rho)} \{ f[\xi, z(\xi), \nabla z(\xi)] - f[x, z_\rho(x), \nabla z(\xi)]\} \rho^{-\nu} \varphi[(\xi - x)/\rho] d\xi$$

$$\leq F_\rho(x) + K\rho + |z(x) - z_\rho(x)| + K \int_{B(x,\rho)} |z(\xi) - z(x)| \rho^{-\nu} \varphi[(\xi - x)/\rho] d\xi ;$$

$$F_\rho(x) = \int_{B(x,\rho)} f[\xi, z(\xi), \nabla z(\xi)] \cdot \rho^{-\nu} \varphi[(\xi - x)/\rho] d\xi$$

Integrating (4.2.6) over D and using Lemmas 2.4.2 and 4.2.6, we get

(4.2.7) $$I(z_\rho, D) \leq I(z, G) + K\rho[|G| + 2 \int_G |\nabla z(x)| dx] .$$

Clearly, the corresponding inequalities hold for the $z_{n\rho}$.

Now, if $z_n \to z$ in $L_1(G)$, $z_{n\rho}$ and $\nabla z_{n\rho}$ tend uniformly on $\overline{D}$ to z_ρ and ∇z_ρ , respectively, and $\| z_n \|^1_1 \leq M$ for all n . Hence for each $\rho < \rho_0$, we have

(4.2.8) $$I(z_\rho , D) = \lim_{n \to \infty} I(z_{n\rho}, D) \leq \liminf_{n \to \infty} I(z_n , G) + K\rho(|G| + M) .$$

Since f satisfies a uniform Lipschitz condition and $z_\rho \to z$ and $\nabla z_\rho \to \nabla z$ in $L_1(D)$, we may let $\rho \to 0$ in (4.2.8) obtaining

$$I(z, D) \leq \liminf_{n \to \infty} I(z_n, G)$$

The result follows from the arbitrariness of D .

THEOREM 4.2.1: Suppose $f(x,z,p)$ is defined and continuous for all (x,z,p) , is convex in p for each (x,z) and $f(x,z,p) \geq f_0(p)$ for all (x,z,p) where $f_0(p)$ is convex and $f_0(p)/|p| \to +\infty$ as $p \to \infty$. Then

$I(z, G)$ <u>is finite or</u> $+\infty$ <u>for each</u> z <u>in</u> $H_1^1(G)$ <u>and is lower semicontinuous with respect to weak convergence in</u> $H_1^1(G)$.

<u>Proof</u>: In order to prove this, it is sufficient to show that $f(x,z,p)$ is the limit of a non-decreasing sequence $f_n(x,z,p)$ each of which has the properties required in Lemma 4.2.8. In order to do this, let $b(x,z;a)$ $(a = \{a_i^\alpha\})$ be chosen so that the function $\varphi(x,z;p;a) \equiv a_i^\alpha p_\alpha^i + b(x,z;a)$ is the unique supporting plane (in p) to f determined by a. By Lemmas 4.2.4 and 4.2.5, $b(x,z;a)$ is continuous in $(x,z;a)$ and $b(x,z;a) \geq b_0(a)$, the corresponding function for f_0. For each a, choose a non-decreasing sequence $b_n(x, z; a)$ of functions, each $\geq b_0(a) - 1$, each satisfying a uniform Lipschitz condition for all (x, z), which converges to $b(x,z;a)$. We then define $\varphi_n(x,z,p;a) = a_i^\alpha p_\alpha^i + b_n(x,z;a)$ and we see that φ_n is a non-decreasing sequence tending to φ for each a, each φ_n satisfying a uniform Lipschitz condition everywhere.

For each n, we define $f_n(x,z,p) = \max \varphi_n(x,z,p;a)$ for all a for which all the a_1^α are rational numbers having numerator and denominator both $\leq n$. Then it is clear that the f_n are non-decreasing and each satisfies a uniform Lipschitz condition. Now, let (x_0,z_0,p_0) and $\varepsilon > 0$ be given. Using Lemma 4.2.5 and the continuity of b, we see that there is a rational $\bar{a}$ such that $\varphi(x_0,z_0,p_0;\bar{a}) > f(x_0,z_0,p_0) - \varepsilon/2$. Clearly $\varphi_n(x_0,z_0,p_0;\bar{a}) > \varphi(x_0,z_0,p_0;\bar{a}) - \frac{\varepsilon}{2}$, for all sufficiently large n, so that $f_n(x_0,z_0,p_0) \longrightarrow f(x_0,z_0,p_0)$.

We now turn to existence theorems. We begin with the following theorem (cf. [ ] and [ ], theorem 8.8 and [ ]):

LEMMA 4.2.9: <u>Suppose</u> $f_0(p)$ <u>is convex in</u> p <u>and</u> $f_0(p)/|p| \longrightarrow +\infty$ <u>as</u> $p \longrightarrow \infty$. <u>Then there is a function</u> $\varphi(\rho) \longrightarrow 0$ <u>as</u> $\rho \longrightarrow 0$ <u>which depends</u>

only on f and M such that if $I_0(z,G) \leq M$, then

$$\int_e |\nabla z(x)|dx \leq \varphi[m(e)] .$$

Proof: For each integer $r \geq 1$, let E_r be the set of x in G where $r - 1 \leq |\nabla z(x)| < r$ and $\nabla z(x)$ exists and let

$$\mathcal{E}_r = \bigcup_{k=r+1}^{\infty} E_k \cup Z , \qquad r = 0, 1, 2, \ldots$$

where Z is the set of measure 0 where $\nabla z(x)$ does not exist. Clearly $\mathcal{E}_0 = G$ and if $r \geq 1$ and $x \in G - \mathcal{E}_r$, then $|\nabla z(x)| < r$. Let α_r be the inf. of $f_0(p)/|p|$ for $|p| \geq r - 1$. Then $\alpha_r \longrightarrow +\infty$ as $r \longrightarrow \infty$. Also

$$\sum_{k=r+1}^{\infty} \alpha_k \cdot (k - 1)\cdot m(E_k) \leq \int_G f_0(\nabla z)dx \leq M$$

From this we see that

$$m(\mathcal{E}_r) \leq \frac{M}{r\cdot \alpha_{r+1}} , \quad \int_{\mathcal{E}_r} |\nabla z|dx \leq \frac{(r+1)M}{r\cdot\alpha_{r+1}}$$

and both $\longrightarrow 0$ as $r \longrightarrow \infty$. So, let e be any measurable subset of G. Let r be the smallest integer such that $M/r\alpha_{r+1} \leq m(e)$. Then

$$\int_e |\nabla z|dx \leq \int_{e-\mathcal{E}_r} |\nabla z|dx + \int_{e\cap\mathcal{E}_r} |\nabla z|\, dx$$

$$\leq \frac{M}{\alpha_{r+1}} + \frac{(r+1)M}{r\cdot\alpha_{r+1}} = \varphi[m(e)]$$

and φ satisfies the conditions.

THEOREM 4.2.2: <u>Suppose</u> f <u>satisfies the hypotheses of Theorem 4.2.1,</u> G <u>is a bounded domain of class</u> C', F <u>is a bounded closed set of functions</u> $z^* \in L_1(\partial G)$, <u>and there is some function</u> $z_1 \in H_1^1(G)$ <u>whose boundary values</u> $\in F$ <u>and for which</u> $I(z_1, G)$ <u>is finite. Then</u> $I(z)$ <u>takes on its minimum among all</u> z <u>in</u> $H_1^1(G)$ <u>having boundary values in</u> F.

<u>Proof</u>: For, let F_1 be the non-empty family of all z in $H_1^1(G)$ for

which z has boundary values in F and $I(z, G) \leq I(z_1, G)$. From Lemma 4.2.9 and Theorems 2.4.2 and 2.4.3 it follows that F_1 is compact with respect to weak convergence in $H_1^1(G)$. The result follows from the lower-semi-continuity.

EXAMPLES: The problem of Plateau reduces to minimizing the Dirichlet integral among all vectors whose boundary values give a representation of the given boundary curve in space. If only an arc of the boundary is prescribed, the remaining part of the boundary being free, we arrive at a type of problem which can be handled using Theorem 4.2.2 and the result of the exercise below.

EXERCISE

Suppose G is of class C', σ is an open set on ∂G. Show that there is a $C(\nu, p, G, \sigma)$ of such that $\| u \|^1_{p,G} \leq C(\| \nabla u \|^0_p + \| u \|^0_{p,\sigma})$.

4.3 Preliminary results on interior differentiability. In this section, we show that the derivatives of the solutions of certain minimum problems are bounded on interior domains. The method used is an adaptation by a student E. R. Buley of a very simple proof of the famous De Giorgi-Nash results [ ] and [ ] due to Maser [ ].

In this section, we restrict ourselves to functions f in (4.1.11) which satisfy the following conditions:

CONDITIONS ON $f(x, z, p)$ in (4.1.11):

(i) f is of class C'' for all (x, z, p); $N = 1$; $\nu > 2$;

(ii) there exist constants k, m, M, M_1, and K with $0 < m \leq M$ and $k \geq 1$ such that

(4.3.1) m $$mV^k - K \leq f(x, z, p) \leq MV^k,$$

(4.3.2) $$\sum [f^2_{p_\alpha} + f^2_{p_\alpha x_\gamma} + f^2_z + f^2_{zx_\gamma}] \leq M_1 V^{2k-1}$$

$$\sum[f_{p_\alpha z}^2 + f_{zz}^2] \leq M_1 V^{2k-2} \quad , \quad V = 1 + z^2 + |p|^2$$

$$m V^{k-1}|\lambda|^2 \leq f_{p_\alpha p_\beta} \lambda_\alpha \lambda_\beta \leq M V^{k-1}|\lambda|^2 \tag{4.3.3}$$

for all λ and all (x, z, p) .

ALTERNATE CONDITIONS ON f: <u>The same as the above except that</u> f <u>is independent of</u> z <u>and</u> $V = 1 + |p|^2$.

The proofs in case f satisfies the alternate conditions are essentially identical with those for the case that f satisfies the original conditions and will be omitted. We notice that $f = V^k$ satisfies either set of conditions and any f above satisfies the conditions of the preceding section. For the remainder of this section, we assume that f satisfies the original conditions above.

THEOREM 4.3.1: <u>Suppose</u> $z \in H_{2k}^1(G)$ <u>and minimizes</u> $I(z)$ <u>among all</u> z <u>with the same boundary values. Then</u>

$$\int_G (\zeta_{,\alpha} f_{p_\alpha} + \zeta f_z)\, dx = 0 \text{ for all } \zeta \in H_{2k0}^1(G) . \tag{4.3.4}$$

<u>Proof</u>: Since $z + \lambda\zeta \in H_{2k}^1$, our hypotheses imply that $I(z + \lambda\zeta) = \varphi(\lambda)$ is bounded for all λ and any $\zeta \in H_{2k,0}^1$. For almost all x ,

$$f[x, z+\lambda\zeta, \nabla z + \lambda\nabla\zeta] = f[x,z,\nabla z] + \lambda\{\zeta_{,\alpha} f_{p_\alpha} + \zeta f_z\}$$

$$+ \lambda^2[A^{\alpha\beta}\zeta_{,\alpha}\zeta_{,\beta} + 2B^\alpha \zeta\zeta_{,\alpha} + C\zeta^2] , \quad \text{where} \tag{4.3.5}$$

$$A^{\alpha\beta}(x) = \int_0^1 f_{p_\alpha p_\beta}[x, z + t\lambda\zeta, \nabla z + t\lambda\nabla\zeta]\, dt , \text{ etc.}$$

The hypotheses imply that

$$\sum[(A^{\alpha\beta})^2 + (B^\alpha)^2 + C^2] \leq \int_0^1 (\nu M^2 + 2M_1)[1 + (z + t\lambda\zeta)^2 + |\nabla z + t\lambda\nabla\zeta|^2]^{k-1} dt \tag{4.3.6}$$

Accordingly the integral of the coefficient of λ^2 in (4.3.5) is uniformly

bounded for $|\lambda| \leq 1$, say, so the result follows.

LEMMA 4.3.1: <u>Suppose</u> $F \quad C^{1}(E_P)$, <u>suppose each</u> $u^p \varepsilon H^1_\lambda(G)$ <u>for some</u> $\lambda \geq 1$, <u>suppose</u> $U(x) = F[u^1(x), \ldots, u^P(x)]$ <u>for</u> $x \varepsilon G$, <u>and suppose that</u> U <u>and the</u> $V_\alpha \varepsilon L\mu(G)$ <u>for some</u> $\mu \geq 1$, $\alpha = 1, \ldots, \nu$, <u>where</u>

$$V_\alpha(x) = \sum_{p=1}^{P} F_{,p}[u^1(x), \ldots, u^P(x)]u^p_{,\alpha}(x) \quad , \quad x \varepsilon G \quad (\text{a.e.}) .$$

<u>Then</u> $U \varepsilon H^1_\mu(G)$ <u>and</u> $U_{,\alpha}(x) = V_\alpha(x)$ <u>almost everywhere</u>.

<u>Proof</u>: From Theorem 2.5. 5, we conclude that each element u^p contains a representative function $\bar{u}$ which is absolutely continuous in x^α along segments in G for almost all x'_α , $\alpha = 1$, ..., ν, and that its partial derivatives $\bar{u}_{,\alpha}$ are representatives of $u_{,\alpha}$. If we set $\bar{U} = F(\bar{u}, \ldots, \bar{u}^P)$, it follows that $\bar{U}$ has the same property and, moreover, its partial derivatives are given by V_α almost everywhere. But this implies that the element $U \varepsilon H^1_\mu(G)$ with $U_{,\alpha} = \quad_\alpha$ a.e.; the proof of this is left to the reader.

THEOREM 4.3.2: <u>Suppose</u> $z \varepsilon H^1_{2k}(G)$ <u>and satisfies</u> (4.3.4). <u>Then, on each domain</u> $D \subset\subset G$, <u>the</u> p_γ <u>and the function</u> $U = V^{k/2} \varepsilon H^1_2(D)$, <u>and the</u> p_γ <u>and</u> U <u>satisfy</u>

$$(4.3.7) \quad \int_D V^{k-1}[\zeta_{,\alpha}(a^\alpha p_{\gamma,\beta} + b^\alpha p_\gamma + e^{\alpha\gamma}) + \zeta(b^\alpha p_{\gamma,\alpha} + cp_\gamma + f^\gamma)]dx = 0$$

$$(4.3.8) \quad \int_D |\nabla U|^2 dx \leq k^2 \int_D V^{k-1}(|\nabla p|^2 + |p|^2)dx \leq Ca^{-2} \int_{D'} V^k dx \text{ , } D \subset D'_\alpha \text{ ,}$$

<u>where the</u> $a^{\alpha\beta}$, b^α , c, $e^{\alpha\gamma}$, <u>and</u> f^γ <u>are bounded and measurable and</u>

$$V^{k-1}(x)a^{\alpha\beta}(x) = f_{p_\alpha p_\beta}[x, z(x), \nabla z(x)] \text{ , } V^{k-1}b^\alpha = f_{p_\alpha z} \text{ , } V^{k-1}c = f_{zz} \text{ ,}$$

$$V^{k-q}e^{\alpha\gamma} = f_{p_\alpha x\gamma} \text{ , } V^{k-q} f^\gamma = f_{zx\gamma} \text{ , } q = 1/2 \text{ .}$$

<u>Proof</u>: We begin by applying the difference quotient procedure of § 3.2. Let $D \subset\subset D' \subset\subset G$ and suppose $\zeta \varepsilon H^1_{2k,0}(G)$ with support in D' , suppose

$D' \subset G_{h_0}$, suppose $1 \leq \gamma \leq \nu$ and e_γ is the unit vector in the x^γ direction, and define

$$\zeta_h(x) = h^{-1}[\zeta(x - he_\gamma) - \zeta(x)] \ , \ z_h(x) = h^{-1}[z(x + he_\gamma) - z(x)] \ , \ 0 < |h| < h_0 \ .$$

Substituting ζ_h for ζ in (4.3.4) and making the obvious changes of variables to eliminate $\zeta(x - he_\gamma)$, etc., we obtain

$$(4.3.9) \quad \int_G A_h(x)\{\zeta_{,\alpha}[a_h^{\alpha\beta} z_{h,\beta} + b_h^\alpha z_h + e_h^{\alpha\gamma} p_h^q] + \zeta[b_h^\alpha z_{h,\alpha} + c_h z_h + f_h^\gamma P_h^q]\}\, dx = 0$$

$$A_h(x) = \int_0^1 [1 + |z(x) + t\Delta z|^2 + |p(x) + t\Delta p|^2]^{k-1}\, dt \ , \ q = 1/2 \ ,$$

$$A_h(x) a_h^{\alpha\beta}(x) = \int_0^1 f_{p_\alpha p_\beta}\, [x + the_\gamma, z(x) + t\Delta z, p(x) + t\Delta p] dt$$

$$A_h b_h^\alpha(x) = \int_0^1 f_{p_\alpha z}\, [\text{same}] dt \ , \ \Delta z = z(x + he_\gamma) - z(x) \ , \text{ etc.}$$

$$P_h(x) = \max_{0 \leq t \leq 1} [1 + |z(x) + t\Delta z|^2 + |p(x) + t\Delta p|^2]$$

and similar formulas holds for the other coefficients. From our hypotheses, it follows that all the coefficients $a_h^{\alpha\beta}$, b_h^α, c_h, $r_h^{\alpha\gamma}$, and f_h^γ are measurable and are bounded independently of h on D' by numbers depending only on ν, m, M, K, M_1, and k. We also have

(4.3.10) $A_h \to A$ in $L_\lambda(D')$ and $z_h \to p_\gamma$ in $L_{2k}(D')$, $\lambda = k/(k-1)$ if $k > 1$; $A = V^{k-1}$; if $k = 1$, $A_h = 1$.

Now if η is Lipschitz and has support in D' and if $0 < |h| < h_0$, we may set

$$(4.3.11) \qquad \zeta = \eta^2 z_h \ , \ \eta = 1 \text{ in } D \ ,$$

in (4.3.9). From our hypotheses, we obtain

$$(4.3.12) \qquad m|\lambda|^2 \leq a_h^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \leq M|\lambda|^2 \ \text{ for } 0 \leq |h| < h_0 \ , \ x \in D' \ .$$

Using the boundedness of the coefficients and Schwarz's inequality, we thus obtain the result that

$$(4.3.13) \quad \int_D A_h(x)|\nabla z_h(x)|^2 dx \leq C \int_{D'} (\eta^2 + |\nabla \eta|^2) A_h(x)[z_h^2 + P_h] dx$$

and the right side is bounded independenly of h since $V^k \varepsilon L_1(G)$.

Since $k \geq 1$ and $z_h \to p_\gamma$ in $L_{2k}(D')$, it follows from (4.3.13) and the fact that $A_h(x) \geq 1$ that $z_h \rightharpoonup p_\gamma$ in $H_2^1(D)$ for a subsequence of $h \to 0$. From (4.3.10) and (4.3.13), it follows that

$$(4.3.14) \quad \zeta_{,\alpha} A_h^q \to \zeta_{,\alpha} A^q \text{ and } A_h^q a_h^{\alpha\beta} z_{h,\beta} \rightharpoonup A^q a^{\alpha\beta} p_{\gamma,\beta} \text{ in } L_2(D), \text{ etc.}$$

for any $\zeta \varepsilon H^1_{2k,0}(D')$ for a further subsequence of $h \to 0$. Accordingly, we see that each $p_\gamma \varepsilon H_2^1(D) \cap L_{2k}(D)$ and, by semicontinuity,

$$(4.3.15) \quad \int_D V^{k-1} |\nabla p|^2 dx \leq Ca^{-2} \int_{D'} V^k dx \text{ if } D \subset D'_a ;$$

this last follows from (4.3.13) by taking $\eta = 1$ on D, $\eta(x) = 1 - 2a^{-1} d(x, D)$ for $0 \leq d(x, D) \leq a/2$, and $\eta(x) = 0$ elsewhere on D'. From the convergence in (4.3.14), we see that the p_γ satisfy (4.3.7). That $U \varepsilon H_2^1(D)$ and (4.3.8) holds follows from Lemma 4.3.1 and (4.3.15)

The proofs of the following two lemmas are left to the reader:

LEMMA 4.3.2: <u>Suppose</u> ζ <u>and</u> $B \varepsilon H_1^1(D)$, $\Lambda(\zeta) \subset D$, <u>and</u> $\zeta B_{,\gamma}$ <u>and</u> $\zeta_{,\gamma} B \varepsilon L_1(D)$ <u>for some</u> γ. <u>Then</u>

$$\int_D \zeta B_{,\gamma} dx = - \int_D \zeta_{,\gamma} B dx$$

LEMMA 4.3.3: <u>Suppose</u> B <u>and</u> $A^\alpha \varepsilon H_1^1(D)$, $\alpha = 1, \ldots, \nu$ <u>and</u>

$$\int_D (A^\alpha \zeta_{,\alpha} + B\zeta) dx = 0 \text{ for all } \zeta \varepsilon \mathrm{Lip}_c(D).$$

<u>Then</u> $A^\alpha_{,\alpha}(x) = B(x)$ <u>almost everywhere on</u> D.

LEMMA 4.3.4: <u>Suppose</u> $A^\alpha \varepsilon H_1^1(D)$, $\alpha = 1, \ldots, \nu$ <u>and suppose</u> $\zeta, \psi, \psi\zeta$, <u>and</u> $\psi^{-1} A^\alpha \varepsilon H_2^L(D)$, <u>suppose</u> $\psi \nabla \zeta$ and $\psi^{-1} \nabla A^\alpha$ <u>all</u> $\varepsilon L_2(D)$, <u>suppose</u> $\psi(x)$

≥ 1 a.e. on D, and suppose $\Lambda(\zeta) \subset D$. Then

$$\int_D (\zeta_{,\alpha} A^\alpha_{,\gamma} - \zeta_{,\gamma} A^\alpha_{,\alpha})\, dx = 0$$

Proof: For each n, let $\psi_n(x) = \psi(x)$ if $\psi(x) < n$ and $\psi_n(x) = n$ on the set E_n where $\psi(x) \geq n$; and define

(4.3.16) $\omega = \psi\zeta$, $C^\alpha = \psi^{-1} A^\alpha$, $\zeta_n = \psi_n^{-1}\omega$, $A^\alpha_n = \psi_n C^\alpha$, $\chi = \ln \psi$

We note that $\nabla\psi_n = 0$ a.e. on E_n and that

(4.3.17) $$\omega \nabla \chi \text{ and } C^\alpha \chi \in L_2(D) .$$

By hypothesis, ω and $C^\alpha \in H^1_2(D)$. Moreover $\nabla\zeta_n = \nabla\zeta$ a.e. on $D - E_n$ and $\nabla\zeta_n = \psi_n^{-1}\nabla\omega$ a.e. on E_n, so $\zeta_n \in H^1_2(D)$. Also

$$\nabla A^\alpha_n = \psi_n(\nabla C^\alpha + C^\alpha \nabla\chi) \text{ a.e. on } D - E_n$$

$$\nabla A^\alpha_n = \psi_n \nabla C^\alpha \text{ a.e. on } E_n$$

so that each $A^\alpha_n \in H^1_2(D)$.

Let us define

(4.3.18) $J_n(x) = \zeta_{n,\alpha} A^\alpha_{n,\gamma} - \zeta_{n,\gamma} A^\alpha_{n\alpha}$, $J(x) = \zeta_{,\alpha} A^\alpha_{,\gamma} - \zeta_{,\gamma} A^\alpha_{,\alpha}$, $J_0 = \omega_{,\alpha} C^\alpha_{,\gamma} - \omega_{,\gamma} C^\alpha_{,\alpha}$

Since $\Lambda(\zeta_n) = \Lambda(\zeta) \subset D$, it is easy to see by approximating to ζ_n and A^α_n by mollified functions that

(4.3.19) $$\int_D J_n(x)\,dx = 0 \text{ for each } n .$$

Moreover $J(x) = J_n(x)$ a.e. on $D - E_n$. On E_n, we see that

(4.3.20)
$$J_n(x) = J_0(x) \text{ a.e.}$$
$$J(x) = J_0(x) + \omega_{,\alpha}(C^\alpha \chi_{,\gamma}) + (\omega\chi_{,\gamma})C^\alpha_{,\alpha} - (\omega_{,\alpha})C^\alpha_{,\gamma} - \omega_{,\gamma}(C^\alpha \chi_{,\alpha}) .$$

The result follows from (4.3.17) - (4.3.20)

LEMMA 4.3.5: Suppose $z \in H^1_{2k}(G)$ and satisfies (4.3.4), suppose $U = V^{k/2}$ and suppose that $U \in L_{2\tau}(D')$, $D' \subset\subset G$, $\tau \geq 1$. Then $w = U^\tau \in H^1_2(D)$ for

each $D \subset\subset D'$ and

$$\int_D |\nabla w|^2 dx \leq C\tau^2 a^{-2} \int_{D'} w^2 dx \quad \text{if} \quad D \quad D'_a, \; C = C(m, M, K, M_1, k, \nu) .$$

Proof: For each L, let $U_L(x) = U(x)$ if $U(x) < L$ and $U_L(x) = L$ on the set E_L where $U(x) \geq L$. Suppose η is defined as in the proof of Theorem 4.3.2 and, for each L, define

(4.3.21) $$\zeta = \eta^2 U_L^{2\tau-2} p_\gamma, \; A^\alpha = f_{p_\alpha}, \; B = f_z, \; \psi = V^{(k-1)/2}$$

Then we notice that $\Lambda(\zeta) \subset D'$ and $\zeta, \psi, \psi\zeta, \psi^{-1}A^\alpha \in H_2^1(D')$, A^α and $B \in H_1^1(D')$, $\zeta_{,\gamma}B$ and $\zeta B_{,\gamma} \in L_1(D')$, and $\psi \nabla \zeta$ and $\psi^{-1}\nabla A^\alpha \in L_2(D')$, and

(4.3.22) $$A^\alpha_{,\gamma} = V^{k-1}(a^{\alpha\beta}p_{\gamma,\beta} + b^\alpha_\gamma + V^q e^{\alpha\gamma}), \; B_{,\gamma} = V^{k-1}(b^\alpha p_{\gamma,\alpha} + cp_\gamma + V^q f^\gamma)$$

So, although ζ, as defined by (4.3.21) is not in $H^1_{2k,0}(D')$, Lemmas 4.3.2, 4.3.3, and 4.3.4 allow us to substitute ζ into (4.3.7), since the left side of (4.3.7) just becomes

$$\int_{D'} (\zeta_{,\alpha}A^\alpha_{,\gamma} + \zeta B_{,\gamma})dx = \int_{D'} (\zeta_{,\alpha}A^\alpha_{,\gamma} - \zeta_{,\gamma}B)\,dx = \int_{D'} (\zeta_{,\alpha}A^\alpha_{,\gamma} - \zeta_{,\gamma}A^\alpha_{,\alpha})dx = 0$$

If we carry out this substitution and use the Schwarz inequality, etc., the condition (4.3.12) for $a^{\alpha\beta}$, etc., the boundedness of U_L and the fact that $\nabla U_L = 0$ a.e. on E_L, we obtain

(4.3.23) $$\int_{D'} \eta^2 U_L^{2\tau-2}[|\nabla U|^2 + (\tau - 1)|\nabla U_L|^2]dx \leq C \int_{D'} U_L^{2\tau-2}[(\eta^2 + |\nabla\eta|^2)U^2 + (\tau - 1)\eta^2 U_L^2]\,dx$$

But since the right side of (4.3.23) is bounded independently of L, we may let $L \to \infty$ in (4.3.23) to obtain

(4.3.24) $$\int_{D'} \eta^2 U^{2\tau-2}|\nabla U|^2 dx \leq C \int_{D'} (\eta^2 + |\nabla\eta|^2)U^{2\tau}dx$$

where C is independent of τ. But (4.3.24) and the definition of η imply the result.

THEOREM 4.3.3: *Suppose* $z \in H^1_{2k}(G)$ *and satisfies* (4.3.4). *Then, on each* $D \subset\subset G$, z *and the* p_γ *are uniformly bounded. In fact, there is a constant* $C = C(\nu, m, M, K, M_1, k)$ *such that*

$$|U(x)|^2 \leq C \cdot |B(x_0, 2R)|^{-1} \int_{B(x_0,2R)} |U(y)|^2 dy , \quad x \in B(x_0, R),\ B(x_0, 2R) \subset G .$$

Proof: To prove this, we combine Lemma 4.3.5 with Theorems 2.5.3 and 2.5.4 which together state that

$$(4.3.25) \quad \left\{ \int_{B_r} \omega^{2s} dx \right\}^{1/s} \leq C_0(\nu) \int_{B_r} (|\nabla w|^2 + r^{-2} w^2) dx , \quad s = \nu/(\nu - 2)$$

if $w \in H^1_2(B_r)$. Let us assume that $B(x_0, 2R) \subset G$, define $B_n = B(x_0, R_n)$ where $R_n = R(1 + 2^{-n})$, $n \geq 0$, and let us define

$$w_n = U^{s^n} \quad \text{so} \quad w_n = w^s_{n-1} , \quad w_0 = U .$$

Then, by applying Lemma 4.3.5 to w_n in order with $D = B_n$ and $D' = B_{n-1}$, we see that each $w_n \in L_2(B_n)$ and to $H^1_2(B_{n+1})$ for each n and that

$$(4.3.26) \quad \left\{ \int_{B_n} w_n^2 dx \right\}^{1/s} \leq C_0 \int_{B_n} (|\nabla w_{n-1}|^2 + R_n^{-2} w^2_{n-1}) dx$$
$$\leq 2C_0 C_1 s^{2n-2} 4^n R^{-2} \int_{B_{n-1}} w^2_{n-1} dx .$$

If we define

$$W_n = \int_{B_n} w_n^2 dx ,$$

(4.3.26) leads to the recurrence relation

$$W_n \leq K_0^s K_1^{ns} \tau^s_{n-1} , \quad K_0 = 2C_0 C_1 s^{-2} R^{-2} , \quad K_1 = 4s^2 , \quad n \geq 1 .$$

This leads to the inequality

$$(4.3.27) \qquad W_n \leq K_0^{\,2+s^2+\dots+s^n} K_1^{ns+(n-1)s^2+\dots+s^n} W_0^{s^n}$$

The result follows by raising both sides of (4.3.27) to the $1/s^n$ power and

taking the limit as $n \longrightarrow \infty$. The multiplier of W_0 in the limit is

$$K_0^{\alpha} K_1^{\beta}, \ \alpha = (1 - s^{-1})^{-1} = \nu/2, \ \beta = (1 - s^{-1})^{-2} = \alpha^2 .$$

EXERCISES

1. Complete the proof of Lemma 4.3.1. 2. Prove Lemmas 4.3.2 and 4.3.3.

4.4. Hölder continuity of the p_γ. In this section, we show that the p_γ are Hölder continuous on domains $D \subset\subset G$. On such domains, we have already seen that they are bounded and each p_γ satisfies an equation of the form

(4.4.1) $\int_D [\zeta_{,\alpha}(a^{\alpha\beta} u_{,\beta} + b^{\alpha} u + e^{\alpha}) + \zeta(c^{\alpha} u_{,\alpha} + du + f)]dx = 0, \ \zeta \in H^1_{20}$

where all the coefficients, e, and f are bounded and measurable and

(4.4.2) $m_2|\lambda|^2 \leq a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \leq M_2|\lambda|^2, \ 0 < m_2 \leq M_2, \ x \in D$.

We shall first discuss the solutions of (4.4.1) in the case where all the b^{α}, C^{α}, d, e^{α}, and f are 0 ; the solutions of such equations are called a-harmonic. We shall present a version of Moser's theory of such equations as presented in [ ]. We first state some technical lemmas involving convex functions on E_1; the reader should recall the theorems stated in Section 4-2 which held for convex functions on any E_N.

LEMMA 4.4.1: Suppose f is convex on E_1. Then

(a) f is Lipschitz on any bounded interval.

(b) Its right-sided and left-sided derivatives exist at each point and are non-decreasing functions.

(c) Its two-sided derivative exists except at $\aleph_0$ points.

(d) If each f_n is convex on E_1, $f_n(u) \longrightarrow f(u)$ for each u, and $f_n'(u_0)$ and $f'(u_0)$ all exist, then $f_n'(u_0) \longrightarrow f'(u_0)$.

(e) If f is non-negative, there exists a non-decreasing sequence $\{f_n\}$

of non-negative convex functions, each Lipschitz over the whole of E_1, such that $f_n(u) \longrightarrow f(u)$ for each u; we may assume that each $f_n(u) = f(u)$ on an interval I_n where $I_n \subset I_{n+1}$.

(f) If f is non-negative and Lipschitz over the whole of E_1 and φ is any non-negative Friedrichs mollifier, the φ-mollified function f_ρ all have the same properties, the Lipschitz constant for f being one for each f_ρ.

LEMMA 4.4.2: Suppose $u \in H^1_\lambda(D)$ $(\lambda \geq 1)$. Then, if c is any constant, $\nabla u(x) = 0$ for almost all x on the set S where $u(x) = c$. If f is convex, $v(x) = f[u(x)]$ on D, $v \in L_\mu(D)$ and the functions V_α defined by

$$V_\alpha(x) = D_{rtf}[u(x)]u_{,\alpha}(x) \in L_\mu(D) ,$$

then v $H^1_\mu(D)$ and $v_{,\alpha}(x) = V_\alpha(x)$ almost everywhere.

Proof: The proof of the second statement is like that of Lemma 4.3.1. If $|S| = 0$, there is nothing to prove. Otherwise, let $[a, b] \subset D$ and choose a representative $\bar{u}$ which is A.C. in x^α on $[a^\alpha, b^\alpha]$ for almost all x'_α. For almost all x'_α, the set $S(x'_\alpha)$ of $x^\alpha \ni (x^\alpha, x'_\alpha) \in S$ is measurable and, for for almost all x^α in $S(x'_\alpha)$, x^α is a limit point of $S(x'_\alpha)$ and the partial derivative $\bar{u}_{,\alpha}(x^\alpha, x'_\alpha)$ exists and so must be 0.

LEMMA 4.4.3: Suppose $u \in L_2(D)$ and is a-harmonic on D and suppose $0 < b \leq r$, $B_{r+b} = B(x_0, r+b) \subset D$, f is a non-negative convex function, $v(x) = f[u(x)]$ on B_{r+b}, and $v \in L_2(B_{r+b})$. Then v $H^1_2(B_r)$ and there is a $C = C(m_2/m_2)$ such that

$$\int_{B_r} |\nabla v|^2 dx \leq Cb^{-2} \int_{B_{r+b}} v^2(x)dx \tag{4.4.3}$$

Proof: First suppose that f is Lipschitz with constant K on the whole of E_1. Using mollifiers (Lemma 4.4.1(f)), we may approximate to f by similar functions f_n of class C^∞ with Lipschitz constant K. Letting

$v_n(x) = f_n[u_n(x)]$, $\zeta = \omega f_n'(u)$ in (4.4.1)($b^\alpha = \dots f = 0$) , we obtain

$$(4.4.4) \qquad \int_{B_{r+a}} (\omega_{,\alpha} a^{\alpha\beta} v_{n,\beta} + \omega f_n'' a^{\alpha\beta} v_{n,\alpha} v_{n,\beta})dx = 0$$

From (4.4.4), we conclude

$$(4.4.5) \qquad \int_{B_{r+b}} \omega_{,\alpha} a^{\alpha\beta} v_{n,\beta}\, dx \le 0 \quad \text{if } \omega \ge 0 .$$

Since $|f_n'| \le K$, $v \varepsilon H_2^1(B_{r'})$ for $r' < r$ so (4.4.5) holds for all ω $H_{20}^1(B_{r+b})$ with compact support. Defining η as was done just before (4.3.24), $\omega = \eta^2 v_n$, we obtain

$$\int_{B_{r+a}} (\eta^2 a^{\alpha\beta} v_{n,\alpha} v_{n,\beta} + 2\eta\eta_{,\alpha} v_n a^{\alpha\beta} v_{n,\beta})dx \le 0$$

from which we conclude (4.4.3) for v_n , using the Schwarz inequality. Now as $n \longrightarrow \infty$, $|\nabla v_n(x)| \le K|\nabla u(x)|$, $|v_n(x)| \le K|u(x)| + C_1$ and $\nabla v_n(x) \longrightarrow \nabla v(x)$ and $v_n(x) \longrightarrow v(x)$ almost everywhere (use Lemma 4.4.1 (d) and 4.4.2), so we may let $n \longrightarrow \infty$ in (4.4.3) and (4.4.5).

If f is not Lipschitz over the whole of E_1, we may approximate to it by such functions f_n as in Lemma 4.4.1 (d). Then $v_n(x) \le v(x)$ for each x , so we may let $n \longrightarrow \infty$ on the right in (4.4.3). But, from $(4.4.3)_n$, it follows that a subsequence of the v_n converge weakly in $H_2^1(B_r)$ to something which must be v . Then (4.4.3) holds for v by lower-semicontinuity.

THEOREM 4.4.1: Suppose u is a-harmonic in B_{2R}, $u \varepsilon L_2(B_{2R})$, f is non-negative and convex, and $v = f(u) \varepsilon L_2(B_{2R})$. Then $v \varepsilon H_2^1(B_r)$, for each $r < 2R$ and there is a $C = C(\nu, M_2/m_2)$ such that

$$|v(x)|^2 \le C|B_{2R}|^{-1} \int_{B_{2R}} v^2(y)dy , \quad x \varepsilon B_R .$$

More generally

$$|v(x)|^2 \le C_1(2R - r)^{-\nu} \int_{B_{2R}} |v(y)|^2 dy \quad \text{for } x \varepsilon B_r , r < 2R .$$

The proof is like that of the last part of Theorem 4.3.3 and is left to the reader.

LEMMA 4.4.4: Suppose f is a function such that $h \equiv -e^{-f}$ is convex and suppose u is a-harmonic on B_{r+b} and $u \in L_2(B_{r+b})$, where $0 < b \leq r$. Then $v = f(u) \varepsilon H_2^1(B_r)$ and

$$\int_{B_r} |\nabla v|^2 dx \leq C b^{-2} r^{\nu}, \quad C = C(\nu, M_2(m_2)) .$$

Proof: Suppose first that $f \varepsilon C^2$ and is Lipschitz for u. Let $\zeta = \eta^2 f'(u)$ in (4.4.1) where η is defined as in Lemma 4.4.3. Then

$$f'' \geq (f')^2$$

since h is convex, so we obtain

$$0 = \int_{B_{r+b}} [2\eta\eta_{,\alpha} a^{\alpha\beta} v_{,\beta} + \eta^2 f''(u) \cdot a^{\alpha\beta} u_{,\alpha} u_{,\beta}] dx$$

$$\geq \int_{B_{r+b}} [2\eta\eta_{,\alpha} a^{\alpha\beta} v_{,\beta} + \eta^2 a^{\alpha\beta} v_{,\alpha} v_{,\beta}] dx$$

from which the result follows from the Schwarz inequality and the definition of η. By approximations (to h) as in the proof of Lemma 4.4.3, we conclude the result as stated.

LEMMA 4.4.5: Suppose $u \varepsilon H_2^1(B_R)$ and that $u(x) = 0$ on a set $S \subset B_R$ such that $|S| \geq c|B_R|$, $c > 0$. Then

$$\int_{B_R} |u(x)|^2 dx \leq C(c, \nu) \cdot R^2 \cdot \int_{B_R} |\nabla u(x)|^2 dx \quad (B_R = B(x_0, R)).$$

Proof: An obvious change of variables shows that we may assume that B_R is the unit sphere $B_1 = B(0, 1)$. The remainder of the proof is left to the reader.

LEMMA 4.4.6: Suppose u is a-harmonic, non-negative, and $u \varepsilon L_2(B_{2R})$ and suppose that $|S| > C_1 |B_{2R}|$, S being the set where $u(x) \geq 1$, $x \varepsilon B_{2R}$. Then

$$u(x) \geq C(\nu, c_1, M_2/m_2), \quad x \in B_{R/2}, \quad 0 < C < 1 .$$

Proof: Define $v(x) = f[u(x)]$, where f is defined by

$$f(u) = \max\{-\log(u+\varepsilon), 0\}, \quad 0 < \varepsilon < 1$$

Then f satisfies the hypotheses of both Theorem 4.4.1 and Lemma 4.4.4. Since $v(x) = 0$ on a set S_ε with $|S_\varepsilon| \geq c_1 |B_{kR}|/2$, for some k with $1 \leq k(c_1, \nu) < 2$, it follows that

$$\int_{B_R} v^2(x)dx \leq \int_{B_{kR}} v^2(x)dx \leq CR^2 \int_{B_{kR}} |\nabla v(x)|^2 dx \leq C_1 |B_R| .$$

Accordingly, by Theorem 4.4.1, we conclude for each ε that

$$|v(x)| \leq C \quad \text{so} \quad u(x) \geq e^{-C} \quad \text{for} \quad x \in B_{R/2} .$$

THEOREM 4.4.2: Suppose that u is a-harmonic on G and that $u \in H_2^1(G)$. Then u satisfies a uniform Hölder condition on any compact subset of G. More specifically,

(4.4.6) $\quad d(u, B_r) \leq C_1 \cdot d(u, B_\delta) \cdot (r/\delta)^{\tau - 1 + \mu_0}, \quad B_r = B(x_0, r) \quad G, \ 0 \leq r \leq \delta .$

(4.4.7) $\quad |u(x) - u(x_0)| \leq C_2 \cdot d(u, B_\delta) \delta^{1-\tau-\mu_0} |x - x_0|^{\mu_0}$ if $|x - x_0| \leq \delta/2$,

$$C_k = C_k(\nu, M_2(m_2), \ \mu_0 = \mu_0(\nu, M_2/m_2), \ 0 < \mu_0 < 1, \ \tau = \nu/2 .$$

Proof: We shall prove (4.4.6); (4.4.7) follows from Theorem 2.5.2.

If u is a-harmonic on $B(x_0, \delta)$ and we define

$$'u(y) = u(x_0 + \delta y), \ 'a^{\alpha\beta}(y) = a^{\alpha}(x_0 + \delta y), \ y \in B(0, 1),$$

we see that $'u$ is a-harmonic on $B(0, 1)$ with the $'a^{\alpha\beta}$ satisfying (4.4.2) with the same m_2 and M_2. From this fact and homogeneity considerations, we may assume that $B(x_0, \delta) = B(0, 1)$ and $d(u, B_1) = 1$. Finally, since $u - \bar{u}$ is a-harmonic, for any constant $\bar{u}$, we may assume that the average of u is zero. Then, from Theorem 2.7.3, it follows that $\|u\|_2^0 \leq C_3(\nu)$. Hence from

Theorem 4.4.1, we conclude that

(4.4.8) $\quad |u(x)| \le C_4$ for $x \in B_{4b}$, $b = 1/8$, $C_4 = C_4(\nu, M_2/m_2)$.

Now, suppose $0 < r \le b$ and suppose that m and M respectively, the essential inf and the essential sup of $u(x)$ on B_{4r} . Choose u_1 so that $|S^{\pm}| \le |B_{4r}|/2$ where S^+ and S^- are, respectively, the sets where $u(x) > u_1$ and $u(x) < u_1$. Then, if we replace the u of Lemma 4.4.6 by $(u - m)/(u_1 - m)$ and $(M - u)/(M - u_1)$ in turn, we conclude that

(4.4.9) $\quad u(x) - m \ge C_5(u_1 - m)$ and $M - u(x) \ge C_5(M - u_1)$ for $x \in B_r$

$$C_5 = C_5(\nu, M_2/m_2) , \; 0 < C_5 < 1 \; .$$

Thus, if we define $\psi(r)$ as the essential oscillation (sup.--inf.) of u on B_r , we conclude that

(4.4.10) $\quad \psi(r) \le h\psi(4r)$, $h = 1 - C_5$, if $r \le b$

Accordingly, using (4.4.8) and (4.4.10), we obtain

$$\psi(r) \le 2h^n C_4 \quad \text{if} \quad 4^{-n}b \le r < 4^{1-n} b, \quad n \ge 1 \; .$$

From this, it follows easily that

(4.4.11) $\quad \psi(r) \le 2C_4(r/b)^{\mu_0} \le C_6 r^{\mu_0}$, $\mu_0 = -\log h/\log 4$, $0 \le r \le b$.

Accordingly, if a set of measure 0 is neglected, $u(x) \longrightarrow u_0$ as $x \longrightarrow 0$, and

(4.4.12) $\quad |u(x) - u_0| \le C_6 |x|^{\mu_0}$ if $0 \le |x| \le b$.

Now, suppose we choose $r \le b/2$ and define $\eta(x) = 1$ on B_r , $\eta(x) = 2 - |x|/r$ on $B_{2r} - B_r$, $\eta(x) = 0$ on $B_1 - B_{2r}$, and set $\zeta = \eta^2 \cdot (u - u_0)$ in (4.4.1) ($b^\alpha = \ldots = f = 0$) . From this we obtain

$$\int_{B_{2r}} [\eta^2 a^{\alpha\beta} u_{,\alpha} u_{,\beta} + 2\eta\eta_{,\alpha}(u - u_0)a^{\alpha\beta} u_{,\beta}]dx = 0$$

from which it follows in the usual way that

$$(4.4.13)\qquad \int_{B_r} |\nabla u|^2 dx \leq C_7 r^{-2} \int_{B_{2r}} (u - u_0)^2 dx \leq C_8 r^{\nu-2+2\mu_0}$$

using (4.4.12). Since $d(u, B_1) = 1$, (4.4.13) holds for $0 \leq r \leq 1$. But this is equivalent to (4.4.6)

Next, we consider (4.4.1) when the coefficients b^α, c^α, and $d \equiv 0$ but e^α and f are not. We handle the term in f by means of a potential using the following theorem.

THEOREM 4.4.3: <u>Suppose</u> $f \in L_2(B_R)$ <u>and</u>

$$(4.4.14)\qquad \int_{B_r} f^2 dx \leq L^2 (r/R)^{\nu-2+2\mu},\ 0 < \mu < 1 .$$

<u>Suppose</u> V <u>is the potential of</u> f. <u>Then</u>

$$(4.4.15)\qquad \int_{B_r} |\nabla V|^2 dx \leq C^2 L^2 R^2 (r/R)^{\nu-2+2\mu},\ C = C(\nu, \mu)$$

<u>Moreover, if</u> $v \in H^1_{20}(B_R)$, <u>then</u>

$$(4.4.16)\qquad \int_{B_R} vf dx = - \int_{B_R} v_{,\alpha} V_{,\alpha} dx$$

<u>Proof</u>: The last statement follows by approximating to v and f by smooth functions. Moreover (4.4.15) holds for $(R/2) \leq r \leq R$, obviously, on account of Theorem 2.6.2. So we assume $r \leq R/2$ and hold it fixed. We define f_1 and f_2 by

$$f(\xi) = f_1(\xi) + f_2(\xi),\ f_1(\xi) = f(\xi) \text{ for } \xi \in B_{2r},\ f_1(\xi) = 0 \text{ otherwise;}$$

and let V_k be the potential of f_k, $k = 1, 2$. As in the proof of Theorem 2.6.2, we find that

$$(4.4.17)\qquad |\nabla V_k(x)| \leq W_k(x),\ W_1(x) = \Gamma_\nu^{-1} \int_{B_{2r}} |\xi - x|^{1-\nu} |f(\xi)| d\xi$$

$$W_2(x) = \Gamma_\nu^{-1} \int_{B_R - B_{2r}} |\xi - x|^{1-\nu} |f(\xi)| d\xi$$

Then, for almost all x, we obtain

$$(4.4.18)\quad W_1^2(x) \leq \Gamma_\nu^{-2} \int_{B_{2r}} |\xi-x|^{1-\nu} d\xi \cdot \int_{B_{2r}} |\xi-x|^{1-\nu} f^2(\xi) d\xi \leq (2r) \cdot \Gamma_\nu^{-1} \int_{B_{2r}} |\xi-x|^{1-\nu} f^2(\xi) d\xi$$

since, for any set S of finite measure, it is obvious that

(4.4.19) $$\Gamma_\nu^{-1} \int_S |\eta|^{1-\nu} d\eta \leq \Gamma_\nu^{-1} \int_{B_\sigma} |\eta|^{1-\nu} d\eta = \sigma \quad \text{where} \quad |B_\sigma| = |S| .$$

Integrating (4.4.18) and using (4.4.19), we obtain

(4.4.20) $$\int_{B_r} W_1^2(x)dx = 2r^2 \int_{B2r} f^2(\xi)d\xi \leq 2r^2L^2(2r/R)^{\nu-2+2\mu}$$

Now, clearly

(4.4.21) $$\tfrac{1}{2}(\xi) \leq |\xi - x| \leq \tfrac{3}{2}|\xi| \quad \text{if} \quad x \in B_r, \ \xi \in B_R - B_{2r} .$$

Let us define

(4.4.22) $$\psi(r) = \int_{B_r} |f(\xi)|d\xi$$

Then, by the Schwarz inequality

(4.4.23) $$\psi(r) \leq \gamma^{*1/2} LR^{1-\nu-\mu} r^{\nu-1+\mu} , \quad 0 \leq r \leq R$$

Then, using (4.4.17), (4.4.21)—(4.4.23), we obtain

$$W_2(x) \leq C_1 \int_{2r}^R \rho^{1-\nu} \psi'(\rho)d\rho \leq C_2L[R^{1-\nu}\psi(R) + (\nu-1)\int_{2r}^R \rho^{-\nu}\psi(\rho)d\rho] \leq C_3LR^{1-\tau}$$

Hence

(4.4.24) $$\int_{B_r} W_2^2(x)dx \leq C_4^2L^2R^{2-\nu}r^\nu \leq C_4^2L^2R^2(r/R)^{\nu-2+2\mu}$$

The result follows from (4.4.20), (4.4.24), and the Minkowski inequality.

THEOREM 4.4.4: <u>Let u be the unique solution in $H^1_{20}(B_R)$ of (4.4.1) with $b^\alpha = c^\alpha = d = 0$ and suppose that e and $f \in L_2(B_R)$ and satisfy (4.4.14). Then there is a $C = C(\nu, M_2/m_2, \mu)$ such that</u>

$$d(u, B_r) \leq C(L_e + C_1RL_f)(r/R)^{\nu-1+\mu}, \quad 0 \leq r \leq R , \ \underline{\text{provided}} \ 0 < \mu < \mu_0 .$$

<u>C_1 being the constant of Theorem 4.4.3 and $d(u, B_r) = \|\nabla u\|^0_{2,r}$.</u>

<u>Proof:</u> If we let V be the potential of f, we see from Theorem 4.4.3 that u satisfies (4.4.1) with $f \equiv 0$ and e^α replaced by

$$E^{a} = e^{a} - V_{,a} \ , \ [\int_{B_r} |E|^2 dx]^{1/2} \leq (L_e + C_1 R L_f)(r/R)^{\tau-1+\mu} \ .$$

So it is sufficient to prove the theorem with $f \equiv 0$ and set $L_e = L$.

To do this, we define

$$\varphi(s) = \sup L^{-1} d(u, B_{bs})$$

for all e satisfying (4.4.14) with R replaced by b, u being the corresponding solution of (4.4.1) in $H^1_{20}(B_b)$. Then, choose any e satisfying (4.4.14), choose r and ρ with $0 < r < \rho \leq R$, and write $u = U + H$ on B_ρ where H is the a-harmonic function $= u$ on ∂B_ρ. Then, since H is a-harmonic and $U = 0$ on ∂B_ρ,

$$I(u, B_\rho) = I(U, B_\rho) + I(H, B_\rho) \ , \ I(v, B_\rho) = \int_{B_\rho} a^{\alpha\beta} v_{,\alpha} v_{,\beta} \, dx \ ,$$

$$d(H, B_\rho) \leq C_2 \, (M_2/m_2) \cdot d(u, B_\rho) \leq C_2 L \varphi(\rho/R)$$

from the definition of φ. Then, if we apply Theorem 4.4.3 and the definition of φ, we obtain

$$d(u, B_r) \leq d(U, B_r) + d(H, B_r) \leq L(\rho/R)^{\tau-1+\mu} \varphi(r/\rho) + C_2 L \varphi(\rho/R) \cdot (r/\rho)^{\tau-1+\mu_0} \ . \tag{4.4.25}$$

since e clearly satisfies (4.4.14) on B_ρ with R replaced by ρ and L by $L(\rho/R)^{\tau-1+\mu}$. Since (4.4.25) holds for all e, etc., we obtain

$$\varphi(s) \leq t^{\tau-1+\mu} \varphi(x/t) + C_2 \varphi(t) \cdot (s/t)^{\tau-1+\mu_0} \ , \ 0 < s < t \leq 1 \ , \tag{4.4.26}$$

where we set $s = r/R$, $t = \rho/R$.

Now, it follows easily by setting $v = u$ in (4.4.1) that

$$\varphi(1) \leq M_2/m_2 \ .$$

Next, choose σ with $0 < \sigma < 1$. Then, clearly

$$\varphi(s) \leq S_0 s^{\tau-1+\mu} \quad \text{for } \sigma \leq s \leq 1 \quad \text{if } S_0 = \varphi(1) \cdot \sigma^{1-\tau-\mu}$$

Applying (4.4.26) with $\sigma^2 \leq s \leq \sigma$ and $t = \sigma^{-1}s$, we obtain

(4.4.27) $\varphi(s) \leq S_1 s^{\sigma-1+\mu}$ for $\sigma^2 \leq s \leq 1$, $S_1 = S_0(1 + C_2\omega)$, $\omega = \sigma^{\mu_0-\mu}$

since $S_1 \geq S_0$. Applying (4.4.26) with $\sigma^4 \leq s \leq \sigma^2$, $t = \sigma^{-2}s$, we obtain

$$\varphi(s) \leq S_2 s^{\tau-1+\mu},\ \sigma^4 \leq s \leq 1\ ,\ S_2 = S_0(1 + C_2\omega)(1 + C_2\omega^2)$$

By repeating the process, we find that

$$\varphi(s) \leq Ss^{\tau-1+\mu},\ S = S_0(1 + C_2\omega)(1 + C_2\,\omega^2)(1 + C_2\omega^4)\ldots,\ 0 < s \leq 1\ .$$

We now indicate how to show that the solutions in $H_2^1(D)$ of the general equations (4.4.1) are Hölder continuous on domains $\Delta \subset\subset D$. We choose $B_R = B(x_0, R) \subset D$ and write $u = u_R + H_R$ on B_R , where H_R is the a-harmonic function $= u$ on ∂B_R so that

$$u_R - T_R u_R = w_R \quad \text{on} \quad B_R$$

where $T_R u_R = U_R$ and U_R and w_R are the solutions in $H_{20}^1(B_R)$ of

$$\int_{B_R} [\zeta_{,\alpha}(a^{\alpha\beta}U_{R,\beta} + b^\alpha u_R) + \zeta(c^\alpha u_{R,\alpha} + du_R)]dx = 0$$

$$\int_{B_R} [\zeta_{,\alpha}(a^{\alpha\beta}w_{R,\beta} + b^\alpha H_R + e^\alpha) + \zeta(c^\alpha H_{R,\alpha} + dH_R + f)]dx = 0\ ,$$

respectively. It is then possible to set up the spaces $H_{2,\mu}^1(B_R)$ of functions $u \in H_2^1(B_R)$ with norm $\|u\|_{2,\mu}^1$ defined by

$$\|u\|_{2,\mu}^1 = \max\,[\,\|u\|_2^1,\ \sup(r/R)^{1-\tau-\mu}d(u,\ B_r)]\ ,\ 0 < \mu < \mu_0$$

From Theorem 4.4.2, it follows that if $H_R \in H_2^1(B_R)$, then $H_R \in H_{2,\mu_0}^1(B_R)$ with $\|H\|_{2,\mu_0}^1 \leq C_1 \cdot \|H\|_2^1$, C_1 being the constant of that theorem. Then the reader can prove that the norm of $T_R \leq 1/2$ if $R \leq R_2$. Then it follows that any solution of (4.4.1) satisfies a condition

$$d(u,\ B_r) \leq L(r/R)^{\tau-1+\mu}\ ,\ 0 \leq r \leq R\ ,\ 0 < R \leq R_2$$

whenever $B_R = B(x_0, R) \subset D$. From that, it follows, using Theorem 2.5.2, that

u is Hölder continuous on compact subsets of D.

Thus, it follows that the p_γ are Hölder continuous on compact subsets of G. Hence it follows that the coefficients $f_{p_\alpha p_\beta}$, etc., in (4.3.9) are Hölder continuous, so that the p_γ satisfy equation (4.4.1) in which the coefficients are Hölder continuous. Then, from the results of § 3.6, it follows that the $p_\gamma \in C^{1+\mu}(D)$ for each $D \subset\subset G$; in other words, the function $z \in C^{2+\mu}(D)$ for such D and is easily seen to satisfy Euler's equation which, when written out has the form

$$(4.4.28) \qquad \begin{aligned} & A^{\alpha\beta}(x,z,\nabla z)z_{,\alpha\beta} = D(x,z,\nabla z) \\ & A^{\alpha\beta}(x,z,p) = f_{p_\alpha p_\beta}(x,z,p), \quad D = f_z - p^\alpha f_{zp_\alpha} - f_{p_\alpha x^\alpha} . \end{aligned}$$

In case higher derivatives of f are Hölder continuous on bounded parts of (x,z,p)-space, the difference quotient procedure can be applied to equations (4.4.28) and the results of § 3.4 used repeatedly to show that the corresponding higher derivatives of z are Hölder-continuous on interior domains. We leave the carrying out of this program to the reader.

EXERCISES

1. Prove Theorem 4.4.1.
2. Prove Lemma 4.4.5.
3. Show that there is an $R_2 > 0$ which depends only on ν, M_2/m_2, μ, and the bounds of the b^α, c^α, and d such that the norm of T_R in $H^1_{2,\mu}(B_R)$ $\leq 1/2$ if $0 < R \leq R_2$.
4. Show that if the $A^{\alpha\beta}$ and $D \in C^{1+\mu}$ on any bounded part of (x,z,p) space, then any function $z \in H^1_{2k}(G)$ which satisfies (4.3.4) $\in C^{3+\mu}(D)$ on each domain $D \subset\subset G$; f is supposed to satisfy one of the sets of conditions in § 4.3.

CHAPTER 5

ELLIPTIC EQUATIONS OF HIGHER ORDER

5.1. Elliptic and strongly elliptic equations. In this chapter, we shall consider equations of the form

(5.1.1) $$Lu + \lambda u = f \, , \; Lu = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u$$

where

(5.1.2) $$D^\alpha = D_1^{\alpha_1} \dots D_\nu^{\alpha_\nu} , \text{ and } D_\beta = i^{-1} \partial / \partial x^\beta \, .$$

We shall allow the coefficients and functions to be complex valued.

DEFINITION: For each x, the operator L is a polynomial in D. We define the characteristic polynomial

(5.1.3) $$L'(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \, , \; \xi^\alpha = (\xi^1)^{\alpha_1} \dots (\xi^\nu)^{\alpha_\nu}$$

and will define the principal part of the operator L by

(5.1.4) $$L'(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha \, .$$

DEFINITION: The equation (5.1.1) is said to be elliptic in a domain D iff the characteristic polynomial is never zero for any x in D and any real $\xi \neq 0$; it is uniformly elliptic iff that polynomial is bounded above and below in absolute value for $|\xi| = 1$ and x in D.

DEFINITION: The equation (5.1.1) is said to be strongly elliptic in D iff there is a couplex-valued non-zero function $p(x)$ such that

(5.1.5) $$\mathcal{R}[p(x) L'(x, \xi)] > 0 \text{ for real } \xi \neq 0 \, .$$

REMARK: If the coefficients of the principal part of L are real, then L is strongly elliptic iff it is elliptic. This is not true if the coefficients are complex.

THEOREM 5.1.1: Suppose D is a set, the operator L in (5.1.1) is elliptic (i.e., the equation is) on D, and $\nu > 2$. Then m is even.

<u>Moreover, if $x_0 \varepsilon D$ and ξ_0 and ξ_0' are linearly independent, the polynomial $L'(x_0, \xi_0 + z\xi_0')$ in z has the same number of roots having positive imaginary parts as those having negative imaginary parts.</u>

<u>Proof</u>: We notice first that if z_0 is a root of $L(x_0, \xi_0 + z\xi_0')$, then $-z_0$ is one of $L(x_0, \xi_0 - z\xi_0')$. Since $\nu > 2$, there is a real continuous vector function $\xi(t)$ on $[0, 1]$ such that $\xi'(0) = \xi_0$ and $\xi'(1) = -\xi_0'$ and $\xi(t)$ and ξ_0 are always linearly independent. The results follow since the roots of $L[x_0, \xi_0 + z\xi'(t)]$ vary continuously with t and no root is real for $0 \leq t \leq 1$.

REMARK: That this is not true for $\nu = 2$ is seen by considering the operator

$$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Moreover, it is seen that any function

$$w = (R^2 - z\bar{z})\, f(z)$$

is a solution of $\partial^2 w/\partial \bar{z}^2 = 0$ if $f(z)$ is analytic; each such function vanishes on $\partial B(0, R)$!

5.2. <u>The existence theory for the Dirichlet problem for a strongly elliptic equation.</u> In this section, we discuss the existence theory for the equation (5.1.1) where L is strongly elliptic and of order $2m$ and u and its first $m - 1$ normal derivatives are prescribed on ∂G if this is sufficiently smooth. We assume that the factor p in (5.1.5) is absorbed into L so (5.1.5) holds iwith $p \equiv 1$. By subtracting off a function having these boundary values, we may assume that our desired solution $u \varepsilon H_{20}^m(G)$, in which case we need make no smoothness assumptions on G.

The developments parallel those of §§ 3.1-3.3. If u is such a solution and $v \varepsilon C_c^m(G)$, it follows that

(5.2.1) $$\int_G \bar{v}(Lu + \lambda u)dx = \int_G \bar{v} f\, dx .$$ If the coefficients a_α with $|\alpha| > m$ ε $C_1^{|\alpha|-m-1}(G)$, then each such term $\bar{v}\, a_\alpha D^\alpha u$ can be integrated by parts $|\alpha| - m$ times to reduce (5.2.1) to a special case of an equation of the form (3.1.4) where $C(u, v)$ is given in (3.1.5) and where

(5.2.2) $$B(u, v) = \int_G \sum_{|\alpha|, |\beta| \leq m} A_{\alpha\beta}(x)\overline{D^\alpha v}\, D^\beta u\, dx, \quad L(v) = \int_G \sum_{|\alpha| \leq m} E_\alpha \overline{D^\alpha v} dx$$

Evidently the integrations by parts can be done in many ways giving rise to different forms $B(u, v)$. However, we have the following lemma:

LEMMA 5.2.1: <u>Suppose</u> L <u>is given by</u> (5.1.1) <u>with</u> m <u>replaced by</u> $2m$ <u>and equation</u> (5.2.2) <u>is obtained from</u> (5.2.1) <u>by integration by parts in any of the ways suggested above. Then</u>

(5.2.3) $$\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha = \sum_{|\alpha|=|\beta|\leq m} A_{\alpha\beta}(x)\xi^\alpha \xi^\beta .$$

<u>Proof</u>: Suppose that the formula (5.2.2) for $B(u, v)$ is obtained from the integral of $\bar{v}\, Lu$ by integration by parts in a certain order. It is clear that, for any x_0 in G, one would obtain the formula

(5.2.4) $$\int_G \bar{v} L'(x_0, D)u dx = \int_G \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x_0)\overline{D^\alpha v}\, D^\beta u\, dx$$

which would hold for any u and $v \in C_c^\infty(G)$. In the process of carrying out the integrations by parts in a given order, one would generate a sequence of forms

(5.2.5) $$\sum_{|\alpha|+|\beta|=2m} A^r_{\alpha\beta}(x_0)\overline{D^\alpha v}\, D^\beta u$$

in which each was obtained from the preceding by a single integration by parts. Suppose we consider a single term in (5.2.5) with $|\alpha| < m$; we can write the integral of that term in the form

(5.2.6) $$\int_G A_r^{\alpha_1 \ldots \alpha_h, \beta_1 \ldots \beta_k}(x_0)\overline{D_{\alpha_1} \ldots D_{\alpha_h} v}\, D_{\beta_1} \ldots D_{\beta_k} u dx \quad (h + k = 2m)$$

in which each α_s and β_t is between 1 and ν and the next integration

by parts is with respect to x^{β_1}. Carrying that out, we see that (5.2.6) equals (note the definition of $\overline{D_\alpha)}$

$$\int_G A_r^{\alpha_1 \cdots \alpha_h \, 1 \cdots \beta_k}(x_0)\, D_{\alpha_1} \cdots D_{\alpha_h} \overline{D_{\beta_1} v\, D_{\beta_2} u \cdots D_{\beta_k} u}\, dx$$

It is clear from this that

$$\sum_{|\alpha|+|\beta|=2m} A^r_{\alpha\beta}(x_0)\, \xi^\alpha \xi^\beta = \sum_{|\alpha|+|\beta|=2m} A^{r+1}_{\alpha\beta}(x_0)\, \xi^\alpha \xi^\beta$$

from which the result follows.

As in § 3.1, then we shall consider equations of the form (3.1.4) in which $C(u, v)$ is given in (3.1.5) and $B(u, v)$ and $L(v)$ are defined in (5.2.2) where we assume:

GENERAL ASSUMPTIONS: The coefficients $A_{\alpha\beta}$ are bounded and measurable on the bounded domain G and those for which $|\alpha| = |\beta| = m$ are continuous on $\overline{G}$ and we assume that

$$(5.2.7) \qquad (1 - h)|\xi|^{2m} \leq \operatorname{Re} \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\, \xi^\alpha \xi^\beta \leq (1 + h)|\xi|^{2m}, \quad 0 < h < 1$$

for all x on $\overline{G}$ and all real ξ.

THEOREM 5.2.1 (Garding's inequality): <u>There are constants M_1 and λ_0, which depend only on ν, m, h, bounds for the coefficients, the moduli of continuity of those $A_{\alpha\beta}$ with $|\alpha| = |\beta| = m$, and the diameter of G, such that</u>

$$|B(u, v)| \leq M_1 \| u \| \| v \| , \quad u, v \in H^m_{20}(G)$$

$$\operatorname{Re} B(u, u) \geq \frac{(1 - h)}{2} \| u \|^2 - \lambda_0 C(u, u) \quad (\|\varphi\| = \|\varphi\|^m_{20})$$

$$(\|\varphi\|^2 = \int_G |\nabla^m \varphi|^2 dx)$$

<u>Proof</u>: It is sufficient to prove this for $u, v \in C^m_c(G)$. The existence of M_1 follows from the form of B and the Poincaré inequality (Theorem 2.1.5). To prove the second inequality, let $\varepsilon > 0$. $\overline{G}$ can be covered by a finite

number of spheres $B(x_i, r_i)$ such that $|A_{\alpha\beta}(x) - A_{\alpha\beta}(x_i)| < \varepsilon$ for $x \in \bar{G} \cap B(x_i, r_i)$ for each i, whenever $|\alpha| = |\beta| = m$. Clearly, there is a sequence $\zeta_1, \ldots, \zeta_S$ in which each $\zeta_s \in C^m(E_\nu)$ and has support in some one sphere $B_i = B(x_i, r_i)$ such that $\zeta_1^2 + \ldots + \zeta_S^2 \equiv 1$ on $\bar{G}$. Then, as in the proof of Theorem 3.3.2, we see that

$$\operatorname{Re} B(u, u) = \operatorname{Re} B'(u, u) + \operatorname{Re} \int_G \sum_{s=1}^{S} \sum_{|\alpha|=|\beta|=m} \zeta_s^2 A_{\alpha\beta}(x) \overline{D^\alpha u} D^\beta u dx$$

$$= \operatorname{Re} B''(u,u) + \operatorname{Re} \int_G \sum_{s=1}^{S} \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x) \overline{D^\alpha u_s} D^\beta u_s dx ,$$

where $B''(u, v)$ is a form like $B(u, v)$ but with $A''_{\alpha\beta} \equiv 0$ when $|\alpha| = |\beta| = m$. But now the last term in (5.2.8)

$$= \operatorname{Re} \int_G \sum_{s=1}^{S} \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x_i) \overline{D^\alpha u_s} D^\beta u_s dx \qquad (\text{supt } \zeta_s \subset B_i)$$

$$(5.2.9) \qquad + \operatorname{Re} \int_G \sum_{s=1}^{S} \sum_{|\alpha|=|\beta|=m} [A_{\alpha\beta}(x) - A_{\alpha\beta}(x_i)] \overline{D^\alpha u_s} D^\beta u_s dx$$

Clearly the absolute value of the second term in (5.2.9)

$$(5.2.10) \qquad \leq C_1(\nu, m) \cdot \varepsilon \sum_{s=1}^{S} \| u_s \|^2$$

If we introduce the Fourier transforms $\hat{u}_s$ of u_s by

$$\hat{u}_s(y) = (2\pi)^{-\nu/2} \int_G e^{-ix \cdot y} u_s(x) dx ,$$

the Plancherel theorem shows that the first integral in (5.2.9)

$$= \int_{-\infty}^{\infty} \sum_{s=1}^{S} [\operatorname{Re} A_{\alpha\beta}(x_i) y^\alpha y^\beta |\hat{u}_s(y)|^2] dy \geq (1 - h) \sum_{s=1}^{S} \| u_s \|^2 .$$

Now, using the fact that $\nabla^m u_s = \nabla^m \zeta_s u = \zeta_s \nabla^m_u + M_s u$, where M_s is of order $m - 1$, and using (5.2.8), (5.2.9), (5.2.10), we obtain

$$\operatorname{Re} B(u, u) \geq (1 - h - C_1 \varepsilon) \| u \|^2 - \operatorname{Re} B'''(u, u)$$

where B''' is a form like B''. Using Theorem 2.7.3 (we may think of $\bar{G} \subset B(x_0, R)$) and the device $|2ab| \leq \eta a^2 + \eta^{-1} b^2$, we obtain the result, since ε may be taken arbitrarily small.

Using Theorem 5.2.1 and Theorem 3.1.2 (the Lemma of Lax and Milgram) one

can prove the analog of Theorem 3.1.4 after first proving that of Theorem 3.1.3, the proofs being similar. Then if the coefficients $A_{\alpha\beta}$ are all Lipschitz, the difference-quotient procedure of § 3.2 can be used to show that any solution $u \in H_2^m(D)$ for each $D \subset\subset G \in H_2^{m+1}(D)$ for each such D if the coefficients E_α in $L(v)$ (see (5.2.2)) are in $H_2^1(D)$ for such D. Instead of doing this the way it was done in § 3.2, it is necessary to carry out the procedure of that section for small neighborhoods mapped on spheres or hemispheres. If G is of class C_1^m, the mappings may be of that class and $B(u, v)$ corresponds to a form in which the coefficients $A_{\alpha\beta}$ are Lipschitz in the new coordinates.

LEMMA 5.2.2: _If the coefficients_ $A_{\alpha\beta}$ _satisfy the general assumptions on_ $\overline{B}_{R_0} = B(x_0, R_0)$, _there exists an_ R_1 _with_ $0 < R_1 \leq R_0$ _which depends on the quantities mentioned in Theorem 5.2.1 such that_

$$\operatorname{Re} B(u, u) \geq \frac{(1 - h)}{2} (\|u\|_R)^2 \qquad (\|u\|_R = \|u\|_{20R}^m) \text{ if } 0 < R < R_1$$

_if_ $u \in H_{20R}^m$. _The same result holds with_ B_R _replaced by_ G_R.

Proof: If we set $G = B_R$ or G_R as the case may be,

$$B(u, u) = B'(u, u) + \int_G \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x_0) \overline{D^\alpha u}\, D^\beta u\, dx$$
$$(5.2.11) \qquad + \int_G \sum_{|\alpha|=|\beta|=m} [A_{\alpha\beta}(x) - A_{\alpha\beta}(x_0)]\, \overline{D^\alpha u}\, D^\beta u\, dx$$

where $A'_{\alpha\beta} \equiv 0$ if $|\alpha|=|\beta|=m$. The last term is dominated by

$$(5.2.12) \qquad \mathcal{E}(R) \cdot \|u\|^2$$

and, as in the proof of the preceding theorem, we see that $B'(u, u)$ is also dominated by (5.2.12) and

$$\operatorname{Re} \int_G \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x_0)\, \overline{D^\alpha u}\, D^\beta u\, dx \geq (1 - h)\|u\|^2$$

THEOREM 5.2.2 (Interior boundedness): _Suppose the hypotheses of Lemma 5.2.2 hold and suppose_ $R \leq R_1$. _Suppose the coefficients_ E_α _in_ $L(v)$ (see (5.2.2))

$\in L_2(B_R)$, suppose $u \in H_2^{m-1}(B_R)$ and $u \in H_2^m(B_r)$ for each $r < R$ and suppose u is a solution of (3.1.4) for each $v \in H_{20}^m$ with compact support in B_R . Then

$$(5.2.13) \qquad \int_{B_r} |\nabla^m u|^2 dx \leq C_1(\| u \|_{2R}^{m-1})^2 + C_2 \sum_{|\alpha| \leq m} (\| E_\alpha \|_{2,R}^0)^2$$

where C_1 and C_2 depend only on the quantities mentioned in Theorem 5.2.1 and on R and r . The same result holds if B_r is replaced by G_r for $r \leq R$ if $\nabla^j u = 0$ along σ_R for $0 \leq j \leq m - 1$.

Proof: Let $\zeta(x) = h[(|x - x_0| - r)/(R - r)]$ where h is of class C^∞ on E_1 , $h(s) = 1$ for $s \leq 1/4$ and $h(s) = 0$ for $s \geq 3/4$, h being monotone. It is sufficient to prove this for $r \geq R/2$; in this case

$$|\nabla^j \zeta(x)| \leq h_j \cdot (R - r)^{-j} , \quad j = 1, 2, \ldots .$$

Then, define

$$v(x) = \zeta^m(x) U(x) , \quad U(x) = \zeta^m(x) u(x) .$$

Then it is easy to verify that U , $v \in H_{20}^m(B_R)$, have support there, and

$$(5.2.14) \qquad D^\alpha v = \zeta^m[D^\alpha U + M^\alpha(\zeta, u)] , \quad \zeta^m D^\beta u = D^\beta U + N^\beta(\zeta, u)$$

where M^α involves derivatives of u of order $< |\alpha|$ and N^β those of order $< |\beta|$; both involve derivatives of ζ . From (5.2.14), we conclude that

$$(5.2.15) \qquad B(u, v) = B(U, U) + B_1(U, u) + B_2(u, u), \quad \|L v\| \leq C_3[\|U\|_{20}^m + \|u\|_2^{m-1}] \|E\|_2^0$$

where B_1 and B_2 involve only derivatives of u of order $\leq m - 1$ and $(\|E\|_2^0)^2$ denotes the second term on the right in (5.2.13). Thus, from the lemma, (5.2.15), and the usual devices we see that

$$\frac{(1 - h)}{4} (\| U \|_{20}^m)^2 - C_4(\| u \|_2^{m-1})^2 \leq C_3[\| U \|_{20}^m + \| u \|_2^{m-1}] \| E \|_2^0$$

from which the theorem follows. The proof for G_R is the same.

The following lemma is useful in proving the next theorem.

LEMMA 5.2.3: If Γ is a cell $\Gamma: a^\alpha \leq x^\alpha \leq b^\alpha$ and $\epsilon > 0$, there is

a constant $C(\nu, m, \varepsilon, \Gamma)$ *such that*

(5.2.16) $$\int_\Gamma \sum_{j=1}^{m-1} \sum_{|\alpha|\le j} |D_\nu^{2m-j} D^\alpha u|^2\, dx \le \varepsilon \int_\Gamma |D_\nu^{2m} u|^2 dx + C\int_\Gamma \sum_{j=0}^{m} \sum_{|\beta|\le 2m-j} \cdot |D_\nu^j D^\beta u|^2 dx$$

for all $u \in H_2^{2m}(\Gamma)$; *here, we assume* $\alpha_\nu = \beta_\nu = 0$.

Proof: We first prove the lemma for the case that u has compact support on some cell Γ and show that C_1 can be chosen so that there is a constant C_2 such that

(5.2.17) $$(\| u \|_{2,\Gamma}^{2m})^2 \le C_2[\varepsilon \int_\Gamma |D_\nu^{2m} u|^2\, dx + C_1 J(u, \Gamma)]$$

where J denotes the last integral in (5.2.16). Letting $I(u, \Gamma)$ denote the first integral in (5.2.16) and taking Fourier transforms, (5.2.16) and (5.2.17) are equivalent, respectively, to

$$\int_{-\infty}^{\infty} \sum_{j=1}^{m-1} \sum_{|\alpha|\le j} [(y^\nu)^{2m-j} y^\alpha]^2 |\hat u(y)|^2 dy$$

$$\le \int_{-\infty}^{\infty} \left\{\varepsilon (y^\nu)^{4m} + C_1 \sum_{j=0}^{m} \sum_{|\beta|\le 2m-j} \cdot [(y^\nu)^j y^\beta]^2\right\} |\hat u(y)|^2 dy = \hat K(\hat u)$$

$$\int_{-\infty}^{\infty} (1+|y|^2)^{2m} |\hat u(y)|^2 dy \le C_2 \hat K(\hat u) ,$$

both of which clearly hold if C_1 and C_2 are large enough.

Now, in general, suppose no such C exists for a given ε . Then there is a sequence $u_n \to u$ in $H_2^{2m}(\Gamma)$ such that $\|u_n\|_{2,\Gamma}^{2m} = 1$ for each n and

(5.2.18) $$I(u_n, \Gamma) > \varepsilon \int_\Gamma |D_\nu^{2m} u_n|^2 dx + nJ(u_n, \Gamma) .$$

Since $I(u_n, \Gamma) \le (\|u_n\|_{2,\Gamma}^{2m})^2$, it follows that $u = 0$. Now, let us extend each u_n to a somewhat larger cell $\Gamma' = [a', b']$ by applying formulas like (2.4.1) across each face in turn. Obviously, we may assume the $u_n \in C^m(\Gamma)$. Moreover, there is a constant C_3 such that

(5.2.19) $$\| D^\beta u_n\|_{2,\Gamma}^0 \le \|D^\beta u_n\|_{2,\Gamma'}^0 \le C_3 \| D^\beta u_n\|_{2,\Gamma'}^0$$

for each particular β with $0 \le |\beta| \le 2m$ and the extended $u_n \to 0$ in Γ'.

Now, let h be the function in the proof of Theorem 5.2.2 and let

$$\zeta(x) = \prod_{\alpha=1}^{\nu} h[(x^\alpha - b^\alpha)/(b'^\alpha - b^\alpha)]\cdot h[a^\alpha - x^\alpha)/(a^\alpha - a'^\alpha)] , \quad U_n(x) = \zeta(x)u_n(x) .$$

Since all the derivatives $D^\beta u_n \longrightarrow 0$ in $L_2(\Gamma')$ if $|\beta| \leq 2m - 1$, we see that

(5.2.20) $I(U_n, \Gamma') \geq I(u_n, \Gamma) > C_3^{-1}\varepsilon \int_{\Gamma'} |D_\nu^{2m} U_n|^2 dx + C_3^{-2} nJ(U_n, \Gamma') - \varepsilon_n$

(5.2.21) $\quad 1 \leq \| U_n \|^{2m}_{2,\Gamma'} \leq C_3(1 + \varepsilon_n) , \lim_{n\to\infty} \varepsilon_n \longrightarrow 0$

But since U_n has compact support and (5.2.21) holds, we see from (5.2.17) that

(5.2.22) $\quad I(U_n, \Gamma'_\varepsilon) > (2^{-1}C_3^{-2}\varepsilon) \int_{\Gamma'} |D_\nu^{2m} U_n|^2 dx + 2^{-1}C_3^{-2} nJ(U_n, \Gamma'), \; n > N_1$

But, from the first paragraph, there is a C_1 such that

(5.2.23) $\quad I(U_n, \Gamma') \leq (2^{-1}D_3^{-2}\varepsilon) \int_{\Gamma'} |D_\nu^{2m} U_n|^2 dx + C_1 J(U_n, \Gamma')$

If $n > N_2$, (5.2.22) and (5.2.23) both hold only if U_n 0, which contradicts (5.2.21).

THEOREM 5.2.3. <u>If</u> G <u>is of class</u> C_1^{2m-1}, <u>all the coefficients</u> $A_{\alpha\beta} \in C_1^{|\alpha|-1}(\bar{G})$, <u>and all the coefficients</u> $E_\alpha \in H_2^{|\alpha|}(G)$, <u>then any solution</u> u <u>in</u> $H^m_{20}(G)$ <u>of</u> (3.1.4) ε $H_2^{2m}(G)$.

<u>Proof</u>: Each point x_0 in G ε a sphere $B(x_0, R_0) \subset G$ and each point x_0 on ∂G is in a boundary neighborhood N which can be mapped in the usual way onto the half-cube $C_{R_0} : D \leq x^\nu \leq R_0$, $|x^\alpha| \leq R_0$ for $\alpha < \nu$ by a map of class C_1^{2m}; in the latter case the new coefficients $A_{\alpha\beta}$ and new E_α satisfy the same differentiability conditions and $N \cap \partial G$ corresponds, say to the face $x^\nu = 0$. So we may carry out the difference quotient procedure on spheres B_R or half-cubes C_R and assume $R \leq R_1$.

If we are given r with $R/2 \leq r < R$, choose $r' = r + \delta$, $r'' = r + 2\delta$, where $R = r + 3\delta$. Let $v \in C_c^m(B_{r'})$, choose γ between 1 and ν and let e_γ denote the unit vector in the x^γ direction and define v_h and u_h by

$$v_h(x) = h^{-1}[v(x - he_j) - v(x)] = -h^{-1} \int_0^h v_{,\gamma}(x - te_\gamma)dt \quad 0 < |h| < \delta$$

(5.2.24) $$u_h(x) = h^{-1}[u(x + he_\gamma) - u(x)] . \qquad x \in B_{r''} .$$

Then proceeding as in § 3.2 and transposing the terms involving $D^\alpha v$ for $|\alpha| < m$ to the right side, we see that (3.1.4) becomes

(5.2.25) $$B_0(u_h, v; B') = L_h(v, B') \quad (B' = B_{r'})$$

$$B_0(u_h, v, B') = \int_{B'} \sum_{\substack{|\alpha|=m \\ |\beta| \leq m}} A_{\alpha\beta} \overline{D^\alpha v}\, D^\beta u_h \, dx$$

(5.2.26) $$L_h(v, B') = \int_{B'} \left\{ \sum_{|\alpha|=m} E_{h\alpha} \overline{D^\alpha v} + \sum_{|\alpha|<m} E_{h\alpha} \overline{D^\alpha v_{,\gamma}} \right\} dx$$

$$E_{h\alpha}(x) = h^{-1}\Delta E_\alpha(x) - h^{-1} \sum_{|\beta| \leq m} D^\beta u(x + he_j)\Delta A_{\alpha\beta}(x) \text{ if } |\alpha| = m$$

(5.2.27) $$E_{h\alpha}(x) = h^{-1} \int_0^h \sum_{|\beta| \leq m} \left\{ A_{\alpha\beta}(x + te_\gamma) D u(x + te_\gamma) - E_\alpha(x + te_\gamma) \right\} dt$$

if $|\alpha| < m$

$$(\Delta \varphi(x) = (x + he_\gamma) - \varphi(x))$$

Now, from the formulas (5.2.25) through (5.2.27), we notice that the u_h and the $E_{h\alpha}$ satisfy the hypotheses of Theorem 5.2.2 uniformly in h for $0 < |h| < \delta$ and x on B' so that $\|\nabla^m u_h\|^0_{2,B}$ is uniformly bounded. Hence, for a subsequence of $h \longrightarrow 0$, $u_h \rightharpoonup$ something in $H_2^m(B)$ which must be $u_{,\gamma}$. As in § 3.2, we conclude that the $u_{,\gamma}$ satisfy the limiting equations which are of the form

$$B_0(u_{,\gamma}, v; B) = L_{1\gamma}(v, B)$$

in which the $E_{1\gamma\alpha}$ with $|\alpha| = m$ involve first derivatives of those $A_{\alpha\beta}$ and E_α with $|\alpha| = m$, the $D^\beta u$ with $|\beta| \leq m$, and certain E_α with $|\alpha| = m - 1$; the $E_{1\gamma\alpha}$ with $|\alpha| < m$ involve the $A_{\gamma\beta}$ and E_γ with $|\gamma| = |\alpha| + 1$ and the $D^\beta u$ with $|\beta| \leq m$. It can be verified that the given differentiability of the coefficients and of the E_α allows the difference-quotient procedure to

be carried out m times to conclude that $u \in H_2^{2m}(B_r)$ for each $r < R$.

In case we are working on C_R and $\nabla^j u = 0$ on γ_R (where $x^\nu = 0$) or $0 \leq j \leq m-1$, the difference-quotient procedure can be carried out in the tangential directions to conclude that every $D^\alpha u$ with $|\alpha| \leq m$ in which no derivative with respect to x^ν is involved $\in H_2^m(C_r)$ for each $r < R$ and vanishes with its derivatives of order $\leq m-1$ on γ_r. On any subdomain $C_{r\eta}$ of C_r where $y^\nu > \eta > 0$, the procedure can be carried out in the x^ν direction also, so that $u \in H_2^{2m}$ on any such domain and satisfies an equation

$$Lu = f \tag{5.2.28}$$

on C_r. On account of the ellipticity, the equation (5.2.28) can be solved for $D_\nu^{2m} u$ in terms of the other derivatives. Thus

$$(\| D_\nu^{2m} u \|_2^0)^2_{r\eta} \leq C[I(u, C_{r\eta}) + J(u, C_{r\eta})]$$

But, using the lemma with $\varepsilon = 1/2C$, we conclude that

$$(\| D_\nu^{2m} u \|_2^0)^2_{r\eta} \leq C\, J(u, C_{r\eta}) \tag{5.2.29}$$

But since $J(u, C_r) < \infty$, we use (5.2.29) and the lemma to conclude that $u \in H_2^{2m}(C_r)$.

By keeping track of the differentiability of the coefficients, it is seen that we have proved the following

THEOREM 5.2.4: <u>If G is of class C_1^{2m-1} and the coefficients a_α with $|\alpha| > m \in C_1^{|\alpha|-m-1}(\bar{G})$ and (5.1.5) holds with $p \equiv 1$, then there is a unique solution u in $H_2^{2m}(G) \cap H_{10}^m(G)$ of (5.1.1) for any λ not in a set $\mathcal{C}$ without limit points in the plane. If $\lambda \in \mathcal{C}$, the usual alternative holds.</u>

5.3. Reduction of the differentiability requirements.

It is now possible to remove the differentiability restrictions on the a_α as in § 3-3.

GENERAL ASSUMPTIONS: We assume that the a_α are bounded and measurable with those for which $|\alpha| = 2m$ continuous on our domain $\bar{G}$ and we also assume that the equation (5.1.1) is uniformly elliptic, at least.

LEMMA 5.3.1: (a) <u>There are constants</u> $C(\nu, m, h)$ <u>and</u> R_1, <u>where</u> R_1 <u>depends on</u> ν, m, h, <u>the bounds for the coefficients, and the moduli of continuity of those</u> a_α <u>with</u> $|\alpha| = 2m$, <u>such that</u>

$$\| u \|^{2m}_{2,R} \leq C \| Lu \|^{0}_{2,R}, \quad 0 < R \leq R_1$$

<u>if</u> $u \in H^{2m}_2(B_R)$ <u>and vanishes near</u> ∂B_R ; <u>here</u>

$$h = \min_{|\xi|=1} |L'(x, \xi)| \quad , \quad \xi \text{ real.}$$

(b) <u>The corresponding result holds on</u> G_R <u>if</u> L <u>is strongly elliptic and</u> (5.1.5) <u>holds with</u> $p \equiv 1$, <u>provided</u> u <u>vanishes near</u> Σ_R <u>and</u> $\nabla^j u = 0$ <u>along</u> σ_R <u>for</u> $0 \leq j \leq m - 1$.

<u>Proof of</u> (a): If $L_0 u = \sum_{|\alpha|=2m} a_\alpha(0) D^\alpha u$, we see that

$$L_0 u = Lu - \sum_{|\alpha|=2m} [a_\alpha(x) - a_\alpha(0)] D^\alpha u - \sum_{|\alpha|<2m} a_\alpha(x) D^\alpha u$$

By taking Fourier transforms, we see that

$$(\| L_0 u \|^0_{2R})^2 = \int_{-\infty}^{\infty} |L_0(y)|^2 |\hat{u}_R(y)|^2 dy \geq h^2 \int_{-\infty}^{\infty} |y|^{4m} |\hat{u}(y)|^2 dy$$

$$= h^2 \| \nabla^{2m} u \|^2_R$$

$$\| \nabla^{2m-j} u \|^0_{2,R} \leq 2^{-j/2} R^j \| \nabla^{2m} u \|^0_{2,R} .$$

The remainder of the proof is like that of Lemma 3.3.1.

Proof of (b): We shall first prove (b) with L replaced by the operator L_0 of part (a) which is now strongly elliptic. Since u vanishes near Σ_R, it follows by successive integrations that

$$\| \nabla^{2m-j} u \|^0_{2,R} \leq C_j R^j \| \nabla^{2m} u \|^0_{2,R}$$

The remainder of the proof is like that of Lemma 3.3.1.

Next, let φ be a mollifier in the variables x'_ν and let u_ρ denote the φ-mollified functions in these variables. By approximating u by C^{2m} functions it is easy to derive formulas for the derivatives of u_ρ which show that if $\rho > 0$, each $D^\gamma u_\rho \in C^{m-1}(\bar{G}_R) \cap H_2^{2m}(G_R)$ if γ is any index-vector with $\gamma_\nu = 0$; moreover, if ρ is small enough, the u_ρ vanish near Σ_R and $\nabla^j u_\rho = 0$ along σ_R if $0 \leq j \leq m - 1$. We prove the result for u_ρ and then let $\rho \longrightarrow 0$.

If $0 < \rho < \rho_0$, we have, by integration by parts

$$\operatorname{Re} \int_{G_R} \sum_\gamma (D^\gamma u_\rho) L_0 (D^\gamma u_\rho) dx = \operatorname{Re} \int_{G_R} \sum_\gamma \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(0) \overline{D^{\alpha+\gamma} u_\rho} D^{\beta+\gamma} u_\rho dx$$

(5.3.1)
$$\geq \int_{G_R} k \sum_{|\alpha|=|\gamma|=m} |D^{\alpha+\gamma} u_\rho|^2 dx = J(u_\rho, G_R)\cdot k, \quad k > 0,$$

to use the notation of Lemma 5.2.3; here the summation is extended over all index sequences γ with $|\gamma| = m$ and $\gamma_\nu = 0$. Integrating the left side of (5.3.1) by parts and applying the Schwarz inequality, we obtain

(5.3.2)
$$J(u_\rho, G_R) \leq C(\nu, m)(\| L_0 u_\rho \|_2^0)^2$$

But, $D_\nu^{2m} u_\rho$ can be solved for in terms of $L_0 u$ and the other derivatives. Thus, by repeating the last paragraph of the proof of Theorem 5.2.3, we arrive at the result.

THEOREM 5.3.1: <u>If</u> G <u>is of class</u> C_1^{2m-1} <u>and</u> L <u>is strongly elliptic with</u> $\rho \equiv 1$ <u>in</u> (5.1.5), <u>there is a constant</u> C, <u>depending only on</u> ν, m, k (<u>strong</u> ellipticity bound), G, <u>the bounds for the coefficients, and the moduli of continuity of those</u> a_α <u>with</u> $|\alpha| = 2m$, <u>such that</u>

$$\| u \|_2^{2m} \leq C[\| Lu \|_2^0 + \| u \|_1^0]$$

The proof parallels that of Theorem 3.3.1; of course the affine transformations are neither necessary nor advantageous.

THEOREM 5.3.2: Under the hypotheses of Theorem 5.3.1, there is a real number λ_0 and a constant C, depending only on the quantities mentioned in that theorem, such that

$$\|u\|_2^{2m} \leq C\|Lu + \lambda u\|_2^0 \text{ if } \lambda \text{ real}, \lambda \geq \lambda_0, u \in H_2^{2m}(G) \cap H_{20}^{m}(G).$$

The proof parallels that of Theorem 3.3.2.

THEOREM 5.3.3: The conclusions of Theorem 5.2.4 hold under the hypotheses of this section.

The proof parallels that of Theorem 3.3.3.

5.4. The fundamental solution of an elliptic equation with constant coefficients. In order to obtain "Schauder estimates," i.e., estimates concerning the Hölder continuity, for the solutions of elliptic equations, it is expedient to have a "fundamental solution" for such an equation with constant coefficients to take the place of the function $K_0(y)$ of Chapters 1, 2, and 3. We assume an equation of the form (5.1.1) with $\lambda = 0$ where L is elliptic, has constant coefficients, is of order $2m$, and $L = L'$.

Let us assume that $f \in C_c^\infty(E\ \ )$ and introduce the Fourier transforms

$$(5.4.1)\qquad \begin{aligned} \hat{u}(y) &= (2\pi)^{-\nu/2} \int_{-\infty}^{\infty} e^{-i(y\cdot x)} u(x)dx \\ \hat{f}(y) &= (2\pi^{-\nu/2} \int_{-\infty}^{\infty} e^{-i(y\cdot \xi)} f(\xi)d\xi \end{aligned}$$

Taking the transform of equation (5.1.1) leads to the equation

$$(5.4.2)\qquad L(y)\hat{u}(y) = \hat{f}(y), \quad L(y) = \sum_{|\alpha|=2m} a_\alpha y^\alpha$$

However, $L(y) \neq 0$ for $y \neq 0$ but is homogeneous of degree $2m$ and so vanishes to that order at the origin. Consequently a rigorous solution of (5.1.1) cannot be given using this method, without some modifications.

Accordingly, we shall use (5.4.2) merely to give a heuristic derivation of a formula for the fundamental solution as follows: Taking the inverse Fourier

transform in (5.4.1) and (5.4.2), we obtain

(5.4.3)
$$u(x) = (2\pi)^{-\nu} \int_{-\infty}^{\infty} e^{iy\cdot x} L^{-1}(y)\left[\int_{-\infty}^{\infty} e^{-iy\cdot\xi} f(\xi)d\xi\right]dy$$
$$= \int_{-\infty}^{\infty} K(x-\xi)f(\xi)d\xi, \text{ where}$$

(5.4.4)
$$K(x) = (2\pi)^{-\nu} \int_{-\infty}^{\infty} e^{iy\cdot x} L^{-1}(y)dy$$

formally. Now, in (5.4.4), we take polar coordinates $y = r\eta$, where $|\eta| = 1$ and (5.4.4) becomes

(5.4.5)
$$K(x) = \int_{|\eta|=1} L^{-1}(\eta)\left[(2\pi)^{-\nu}\int_0^{\infty} r^{\nu-1-2m} e^{ir(x\cdot\eta)}dr\right]d\Sigma(\eta)$$

Now, if $m \geq \nu/2$, the integral in the bracket diverges at $r = 0$, and if $m < \nu/2$, it diverges at ∞. However, it is a function $F[i(x\cdot\eta)]$.

Now, if we write

(5.4.6)
$$\varphi(x) = \int_{|\eta|=1} L^{-1}(\eta)F[i(x\cdot\eta)]d\Sigma(\eta)$$

and $F \in C^{2m}$ and analytic, we notice that

(5.4.7)
$$D^{\alpha}\varphi(x) = \int_{|\eta|=1}\eta^{\alpha}L^{-1}(\eta)F^{|\alpha|}[i(x\cdot\eta)]d\Sigma(\eta),$$
$$L\varphi(x) = \int_{|\eta|=1} F^{(2m)}[i(x\cdot\eta)]d\Sigma(\eta) = \Psi(|x|).$$

If we consider the function

(5.4.8)
$$F(\ ) = (2\pi)^{-\nu}\int_0^{\infty} r^{h-1}e^{rs}dr = f_h(s)\cdot(2\pi)^{-\nu}$$

we see that, if $h \geq 1$, this is analytic for $\mathcal{R}(s) < 0$ and, in fact

(5.4.9)
$$f_h(s) = (-1)^{h-1}(h-1)!\, s^{-h}, \quad f_h'(s) = f_{h+1}(s), \quad \mathcal{R}(s) < 0.$$

If we use (5.4.9) to define f_h for $h \leq 0$, we are lead to defining

$$J_h(s) = \begin{cases} \dfrac{s^h}{h!}[\log(-s) - C_h], & h \geq 0,\ C_0 = 0 \\ (-1)^{1-h}(-h-1)!\,s^h, & h < 0,\ C_h = 1 + \ldots + h^{-1},\ h \geq 1. \end{cases}$$

If we set

$$K(x) = (2\pi)^{-\nu}\int_{\Sigma} L^{-1}(\eta)J_{2m-\nu}[i(x\cdot\eta)]d\Sigma(\eta),$$

the step (5.4.7) could not be carried out, since $J_{-\nu}[i(x\cdot\eta)]$ would not be integrable. So, we choose $q = 0$ or 1 so that $\nu + q$ is even and define

(5.4.10)
$$K^*(x) = -(2\pi i)^{-\nu} i^q \int_\Sigma L^{-1}(\eta) J_{2m+q}[i(x\cdot\eta)] d\Sigma(\eta)$$
$$K(x) = \Delta^{(\nu+q)/2} K^*(x) .$$

Now, if ν is even so $q = 0$, it is clear (cf. 5.4.12 below) that

(5.4.11) $$K^*(-x) = K^*(x) = P(x) \log |x| + Q(x) \quad (\nu \text{ even})$$

where P is a homogeneous polynomial of degree $2m$ and Q is positively homogeneous of degree $2m$. If ν is odd, so $q = 1$, we have

$$K^*(x) = -(2\pi i)^{-\nu} i^{-q} \int_\Sigma L^{-1}(\eta) \frac{i^{2m+q}(x\cdot\eta)^{2m+q}}{(2m+q)!} [\log |x| + \log \frac{\xi\cdot\eta}{i} - C_{2m+q}] d\Sigma(\eta)$$

(5.4.12)
$$= -(2\pi i)^{-\nu}(-1)^m |x|^{2m+q} \int_{\xi\cdot\eta>0} L^{-1}(\eta) \frac{(\xi\cdot\eta)^{2m+q}}{(2m+q)!} (-i\pi) d\Sigma(\eta)$$
$$= -(2\pi i)^{-\nu}(\pi/2) i^{2m-1} |x|^{2m+q} \int_\Sigma L^{-1}(\eta)\cdot[|\xi\cdot\eta|^{2m+q}/(2m+q)!] d\Sigma .$$

In this case K^* is positively homogeneous of degree $2m + q$ and

(5.4.13) $$K^*(-x) = K^*(x) .$$

In either case

(5.4.14) $$LK^*(x) = M(x) = -(2\pi i)^{-\nu} i^{-q} \int_\Sigma J_q[i(x\cdot\eta)] d\Sigma(\eta)$$

and K has homogeneity properties like those of K^* and K also satisfies (5.4.13). We now prove

THEOREM 5.4.1: $$\Delta^{(\nu+q-2)/2} M(x) = K_0(x) = \begin{cases} -(\nu-2)^{-1}\Gamma^{-1}|x|^{2-\nu}, & \nu > 2 . \\ \frac{1}{2\pi} \log |x| & , \nu = 2 . \end{cases}$$

<u>Proof</u>: If $\nu = 2k$ so $q = 0$, then

$$M(x) = (-1)^{k-1}(2\pi)^{-2k} \Gamma_{2k} \log |x| + \text{const.}$$

By computation, we see that

$$\Delta^{k-1} M(x) = 2\cdot(2\pi)^{-2k} \Gamma_{2k} (-1)^{k-2} (k-2)!(k-1)!\cdot 2^{2k-4}|x|^{2-2k}$$

The result follows from the formula

$$\Gamma_\nu = \frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})}, \quad \Gamma_{2k} = \frac{2\pi^k}{(k-1)!} .$$

If $= 2k + 1$, $q = 1$, then

$$M(x) = -(2\pi)^{2k-1}(-1)^{k+1} \int_\Sigma i(x\cdot\eta)[\log|x| + \log\frac{\xi\cdot\eta}{i} - 1]d\Sigma(\eta)$$

$$= (-1)^k(2\pi)^{-2k-1}\, i|x| \int_{\xi\cdot\eta>0} (\xi\cdot\eta)(-i\pi)d\Sigma(\eta)$$

$$= (-1)^k\,\pi\cdot(2\pi)^{-2k-1}\,|x|\cdot \int_0^{\pi/2} \Gamma_{2k}\cos\varphi\cdot\sin^{2k-1}\varphi\, d\varphi$$

$$= (-1)^k\,\pi^{-2k}\cdot 2^{-2k-1}\,\Gamma_{2k}\cdot(2k)^{-1}\,|x| = C\,|x|$$

Then

$$\Delta^{-1+(\nu+q)/2} M(x) = \Delta^k M(x) = C(-1)^{k-1}(2k-2)!(2k)|x|^{1-2k}$$

$$= -2^{-2k}\pi^{-k}\frac{(2k-2)!}{(k-1)!}|x|^{1-2k} = -(2k-1)^{-1}\pi^{-k}\frac{(k-\frac{1}{2})\dots(\frac{1}{2})\cdot(k-1)!}{2(k-1)!}$$

$$= -(2k-1)^{-1}\cdot\frac{\Gamma(k+1/2)}{2\pi^{k+1/2}}|x|^{1-2k} = K_0(x) .$$

LEMMA 5.1: If x_0 <u>is a real vector</u> $\neq 0$, <u>there are functions</u> $c^\gamma_\alpha(x)$, <u>analytic near</u> x_0, <u>such that the transformation</u>

(5.4.15) $$\zeta^\gamma = c^\gamma_\alpha(x)\eta^\alpha$$

<u>is a rotation of axes for each real</u> x <u>near</u> x_0 <u>and such that</u>

$$x\cdot\eta = |x|\zeta^1$$

<u>Proof</u>: Let $\xi^\alpha = x^\alpha/|x|$. By a <u>fixed</u> rotation of axes, we may assume $\xi_0 = (1, 0, \dots, 0)$. If, in (5.4.15) the superscript γ denotes the row, we define

$$c^1_\alpha(x) = \xi^\alpha, \quad \alpha = 1, \dots, \nu$$

and then notice that there is a unique way to complete the matrix $c^\gamma_\alpha(x)$ in such a way that the matrix is orthogonal, $c^\gamma_\alpha(x) = 0$ for $\alpha > \gamma \geq 2$, and each determinant in the upper left hand corner is positive. The c^γ_α are analytic.

THEOREM 5.1.2: K^* <u>and</u> K <u>are analytic and</u>

$$LK(x) = 0 \quad \text{for} \quad x \neq 0 .$$

<u>If</u> $f \in C^0_\mu(\bar{B}_R)$, <u>and</u>

$$u(x) = \int_{B_R} K(x - \xi) f(\xi) d\xi \tag{5.4.16}$$

<u>then</u> $u \in C^{2m}_\mu(\bar{B}_R)$ <u>and</u>

$$Lu(x) = f(x) \ , \quad h_\mu(\nabla^{2m} u) \leq C(m, \nu, h) h_\mu(f)$$

<u>Proof</u>: We shall prove the analyticity for ν odd, using the representation (5.4.12); the proof for ν even is similar. Using the lemma, we may introduce variables ζ by

$$\eta^\alpha = c^\alpha_\gamma(x)\zeta^\gamma = \eta^\alpha[x; \zeta]$$

into (5.4.12) and obtain

$$K^*(x) = C|x|^{2m+1} \int_{\zeta' > 0} L^{-1}[\eta(x; \zeta)](\zeta^1)^{2m+1} d\Sigma(\zeta)$$

which is analytic near any real $x \neq 0$.

Define
$$U(x) = \int_{B_R} K^*(x - \xi) f(\xi) d\xi .$$

Then we note that any $(2m + \nu + q - 1)$st derivative

$$D^{2m+\nu+q-1} U(x) = \int_{B_R} \Gamma(x - \xi) f(\xi) d\xi ,$$

where Γ satisfies the hypotheses of Theorem 1.5.4. Hence $U \in C^{2m+\nu+q}_\mu(\bar{B}_R)$ and $u \in C^{2m}_\mu(\bar{B}_R)$ with $h_\mu(\nabla^{2m} u) \leq C h_\mu(f)$, since

$$\Delta^{(\nu+q)/2} U(x) = u(x) .$$

Now, from 5.4.14 and Theorem 5.1.1, it follows that

$$\Delta^{-1+(\nu+q)/2} LU(x) = \int_{B_R} K_0(x - \xi) f(\xi) d\xi$$

so that (corollary to Theorem 1.5.4)

$$\Delta^{(\nu+q/2} LU(x) = L\Delta^{(\nu+q)/2} U(x) = Lu(x) = f(x) .$$

5.5. <u>Hölder continuity and</u> L_p <u>bounds for the solutions of elliptic</u>

equations in the interior. The developments in this section parallel those of § 3.4. We state the following theorem without proof:

THEOREM 5.5.1 (Calderon-Zygmund theorem): Suppose $J \in C^k(E_\nu - \{0\})$ and is homogeneous of degree $-\nu$ with

(5.5.1) $$\int_{\partial B(0,\,1)} J(\eta)\,d\Sigma(\eta) = 0$$

Then there is a constant $C(p)$ such that if $f \in L_p(E_\nu)$, $1 < p < \infty$, and

(5.5.2) $$u(x) = \int_{-\infty}^{\infty} J(x - \xi) f(\xi)\, d\xi,$$

then u is defined as Cauchy principal value almost everywhere, $u \in L_p(E_\nu)$, and

$$\| u \|_p^0 \leq C(p) \| f \|_p^0$$

Now, we proceed as in § 3.4. Suppose that u is a solution of (5.1.1) with $\lambda = 0$ on B_R where the coefficients satisfy the assumptions of § 5.3. We let L_0 be the operator of Lemma 5.3.1 and set

(5.5.3) $$u = u_R + H_R, \quad u_R = P_R(L_0 u)$$

(5.5.4) $$U_R = P_R(f) \text{ iff } U_R(x) = \int_{B_R} K(x - \xi) f(\xi)\, d$$

For u_R, we have the equation

$$u_R - T_R u_R = v_R, \quad T_R u_R = P_R[(L - L_0) u_R],$$

$$v_R = P_R(Lu) + T_R H_R; \text{ also } L_0 H_R(x) = 0 \text{ on } B_R.$$

For $1 < p < \infty$ and $0 < \mu < 1$, we define the space ${}^*C_p^{2m+\mu}$ to consist of all functions $u \in H_p^{2m}(B_R)$ such that $u \in C_\mu^{2m}(B_r)$ for each $r < R$ with

$${}^*|||u|||_{p,R}^{2m+\mu} = \max\,[\,{}'\|u\|_{p,R}^{2m},\ \sup\,(R - r)^{\tau+\mu} h_\mu(\nabla^{2m} u, B_r)]$$

$${}'\| u \|_{p,R}^{2m} = \sum_{j=0}^{2m} R^{-j} \|\nabla^{2m-j} u\|_p^0 \quad (\tau = \nu/p)$$

We define the auxiliary space ${}^*C_p^\mu$ to consist of f, etc., such that

$${}^*||| f |||_{p,R}^\mu = \max\,[\| f \|_{p,R}^0,\ \sup\,(R - r)^{\tau+\mu} h_\mu(f, B_r)]$$

Then from the Calderon-Zygmund theorem and a proof like that of Theorem 3.4.2,

we obtain

THEOREM 5.5.2: *The transformation $u = P_R f$ is a bounded operator from $L_p(B_R)$ to $H_p^{2m}(B_R)$ and from ${}^*C_p^{\mu}(B_R)$ to ${}^*C_p^{2m+\mu}(B_R)$ with bound independent of R in each case.*

LEMMA 5.5.1 (Interpolation lemma): *Suppose μ is a non-negative measure over the set E and $f \in L_p(E, \mu)$ and $L_q(E, \mu)$ where $p < q$. Then $f \in L_r(E, \mu)$ for each r with $p < r < q$ and*

$$\| f \|_r \leq (\| f \|_q)^{q(r-p)/r(q-p)} \cdot (\| f \|_p)^{p(q-r)/r(q-p)}$$

The result holds if $q = +\infty$ if we interpret $f \in L_q(E, \mu)$ to mean that f is essentially bounded on E.

Proof: For each n, let E_n be the set of points x in E where $n^{-1} \leq f(x) \leq n$. Then $\mu(E_n) < \infty$ and the function φ_n defined by

$$\varphi_n(z) = \int_{E_n} |f_n(x)|^z d\mu , \quad f_n(x) = f(x) \text{ on } E_n \text{ and } 0 \text{ on } E - E_n$$

is analytic for all z and is bounded in any strip $a \leq \operatorname{Re} z \leq b$. Let us first suppose that $q < +\infty$. Then

$$|\varphi_n(p + i\eta)| \leq \|f_n\|_p^p , \quad |\varphi_n(q + i\eta)| \leq \| f_n\|_q^q$$

Hence, since $\log |\varphi_n(z)|$ is subharmonic and bounded above in the strip $p \leq \operatorname{Re} z \leq q$, we conclude that

$$\log |\varphi_n(r)| \leq \frac{r-p}{q-p} q \log \| f_n\|_q + \frac{q-r}{q-p} \cdot p \log \| f_n\|_p$$

from which the first result for f_n follows; the last result for f_n then follows by letting $q \longrightarrow +\infty$. The results for f follow by letting $n \longrightarrow \infty$.

THEOREM 5.5.3: *If $H_R \in H_p^{2m}(B_R)$ for some p, $1 < p < \infty$, and $L_0 H_R = 0$ on B_R, then H_R is analytic in B_R and $H_R \in H_q^{2m}(B_r)$ and to ${}^*C_p^{2m+\mu}(B_R)$ for each $q \geq p$, and each $r < R$*

$$'\| H_r \|^{2m}_{q,r} \leq C(\nu, m, h, p, q)\ '\| H_R \|^{2m}_{p,R} \cdot (R - r)^{\lambda}, \quad \lambda = -(\nu/p) + (\nu/q)$$

$$^{*}\||| H_R |||^{2m+\mu}_{p,R} \leq C(\nu, m, h, p, q, \mu)\ \| H_R \|^{2m}_{p,R}$$

Proof: First extend H_R to be in $H^{2m}_{2,0}(B_{2R})$, have support in B_{2R}, and such that (Theorem 2.4.1)

$$'\| H_R \|^{2m}_{p,2R} \leq C(\nu, m, p)\ '\| H_R \|^{2m}_{p,R}$$

Now approximate to H_R in $H^{2m}_{p,}(B_{2R})$ by functions $U \in C^{2m}_c(B_{2R})$. For each such U, let $x_0 \in B_{2R}$ and define

$$v_\rho(x) = \rho^{-\nu} \int_{B(x,\rho)} \varphi\left(\frac{\xi - x}{\rho}\right) K(\xi - x_0)\, d\xi = \rho^{-\nu} \int_{B(0,\rho)} \varphi\left(\frac{\eta}{\rho}\right) K[x - (x_0 - \eta)]\, d\eta$$

$$= \rho^{-\nu} \int_{B(x_0,\rho)} \varphi\left(\frac{x_0 - \zeta}{\rho}\right) K(x - \zeta)\, d\zeta, \quad B(x_0, \rho) \subset B_{2R}.$$

We note that

(5.5.5) $$Lv_\rho(x) = \rho^{-\nu} \varphi\left(\frac{x_0 - x}{\rho}\right)$$

Since L is self-adjoint, we see that

(5.5.6) $$\int_{B_{2R}} [v_\rho LU - ULv_\rho]\, dx = 0$$

Letting $\rho \to 0$ we obtain

(5.5.7) $$U(x_0) = \int_{B_{2R}} K(x_0 - x) LU(x)\, dx$$

Then (5.5.7) holds for H_R for $x_0 \in B_R$. Thus H_R is analytic in B_R.

From (5.5.7) for H_R and the fact that $LH_R \in L_p(B_{2R})$ and is 0 in B_R, we find that

(5.5.8) $$|\nabla^{2m} H_R(x) \leq C \int_{B_{2R}} |\xi - x|^{-\nu} |LH_R(\xi)\, d\xi$$

Applying the Hölder inequality to (5.5.8), we find that

(5.5.9) $$|\nabla^{2m} H_R(x)| \leq C(R - |x|)^{-\nu/p}\ '\| H_R \|^{2m}_p, \quad |x| \leq r.$$

Then the first conclusion follows from the interpolation lemma. Moreover, by differentiating once more, we obtain

$$|\nabla^{2m+1} H_R(x)| \leq CK \cdot (R - |x|)^{-1-\nu/p} \leq CK(R - r)^{-\mu-\nu/p}(r - |x|)^{\mu-1}$$

from which the last result follows, using Theorem 1.5.2.

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